A review of isogeny based cryptography

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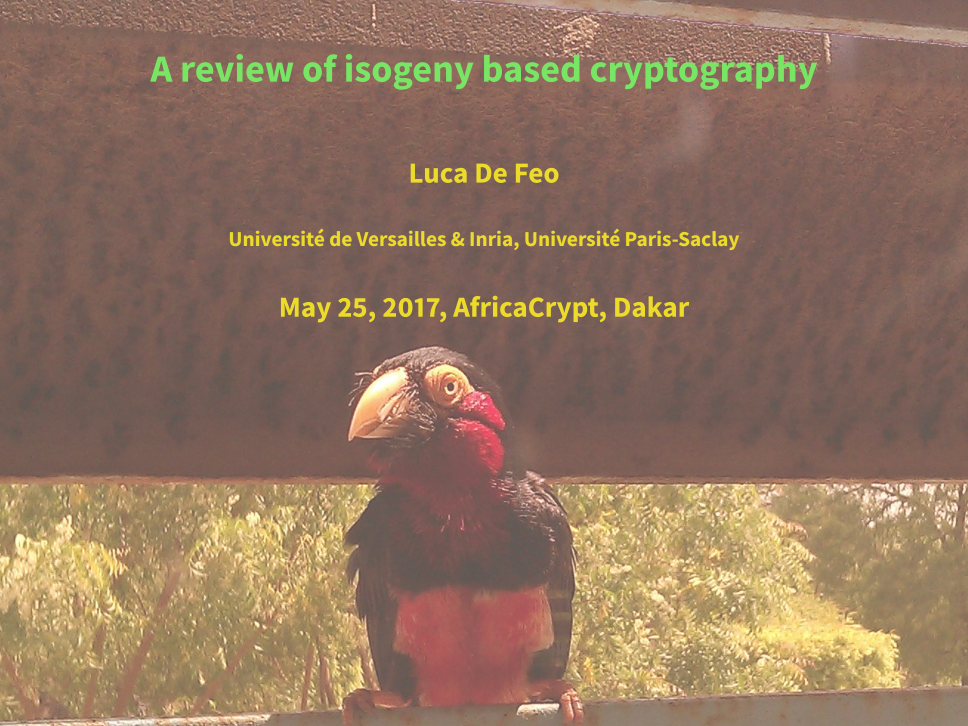
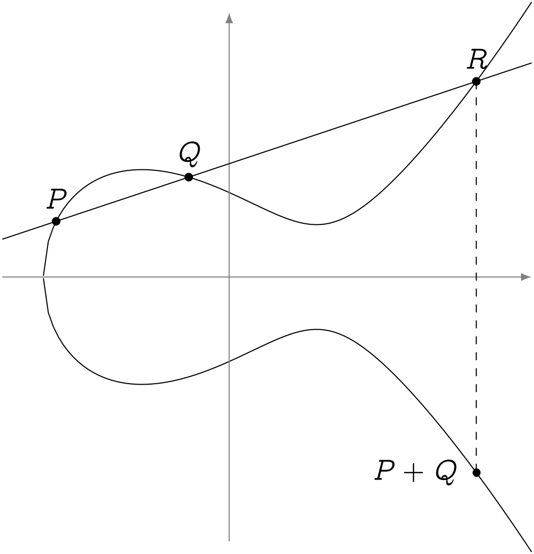


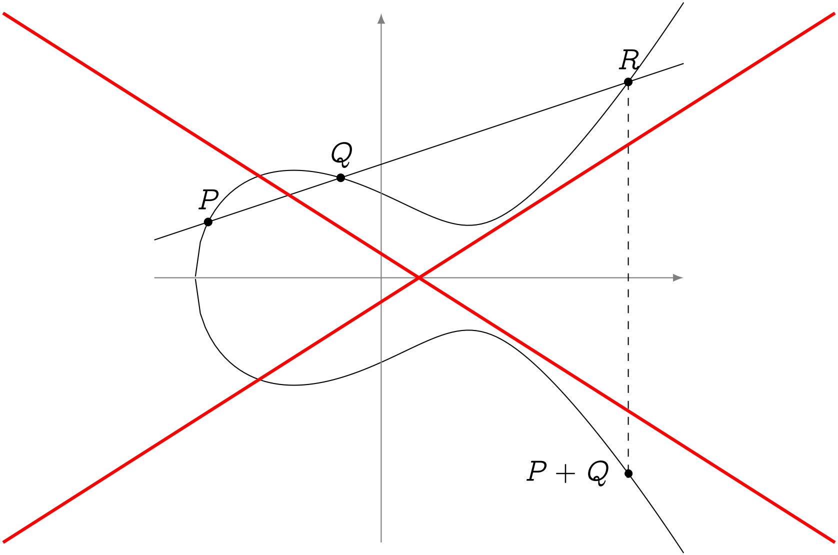
Elliptic curves

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve…



Elliptic curves

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve…forget it!



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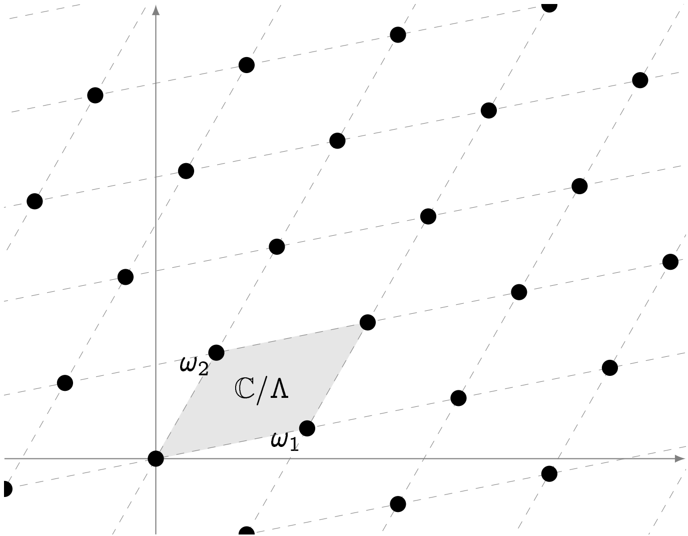
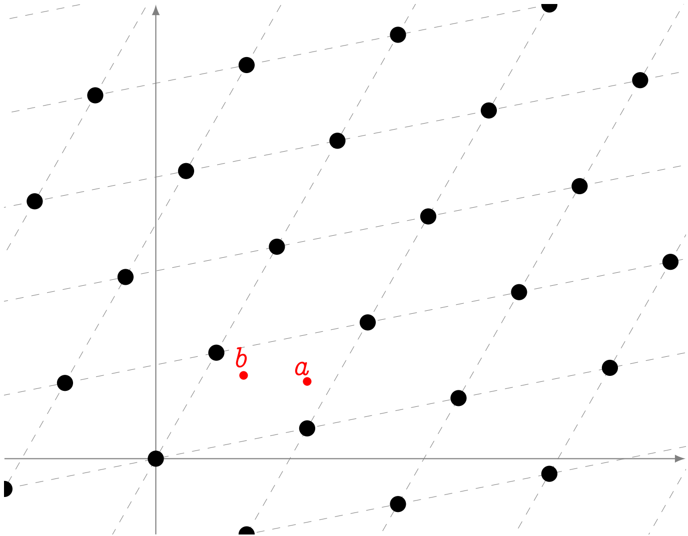


Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent complex numbers. Set

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$$

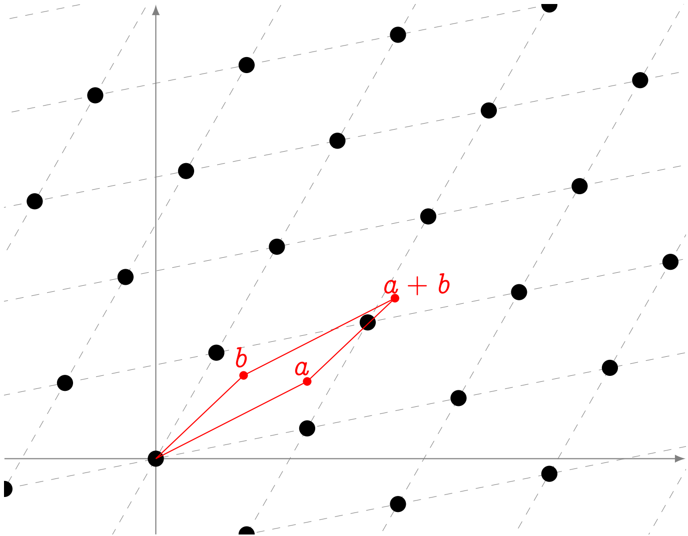
$\mathbb{C}/\Lambda$ is an elliptic curve.

Elliptic curves



Addition law induced by addition on $\mathbb{C}$.

Elliptic curves



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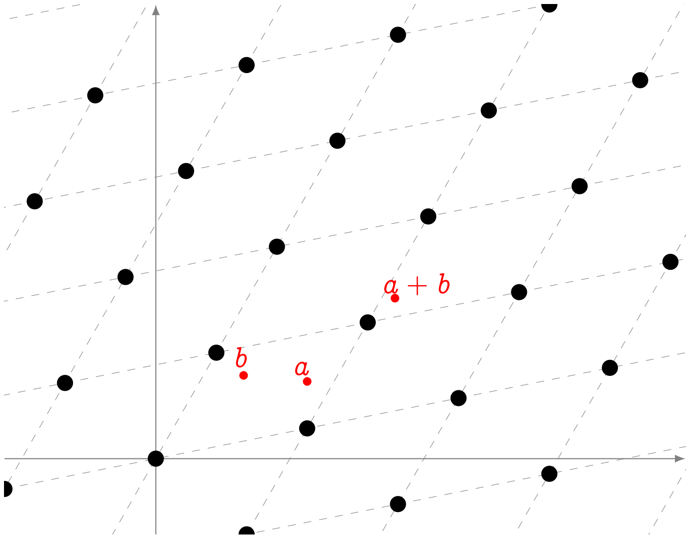

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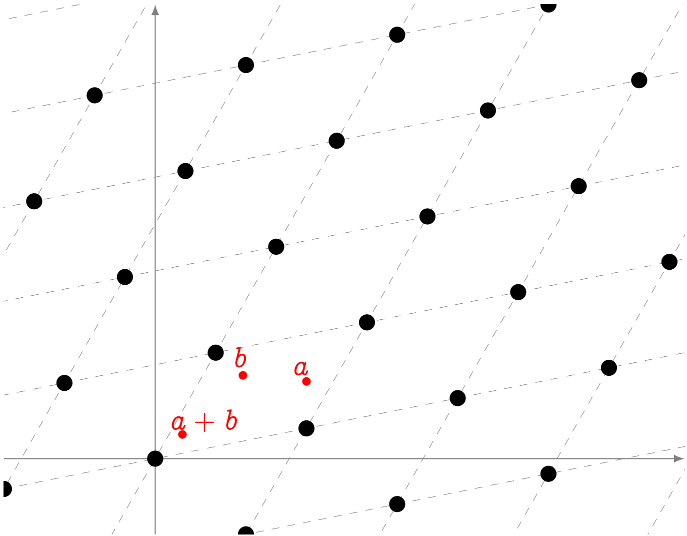

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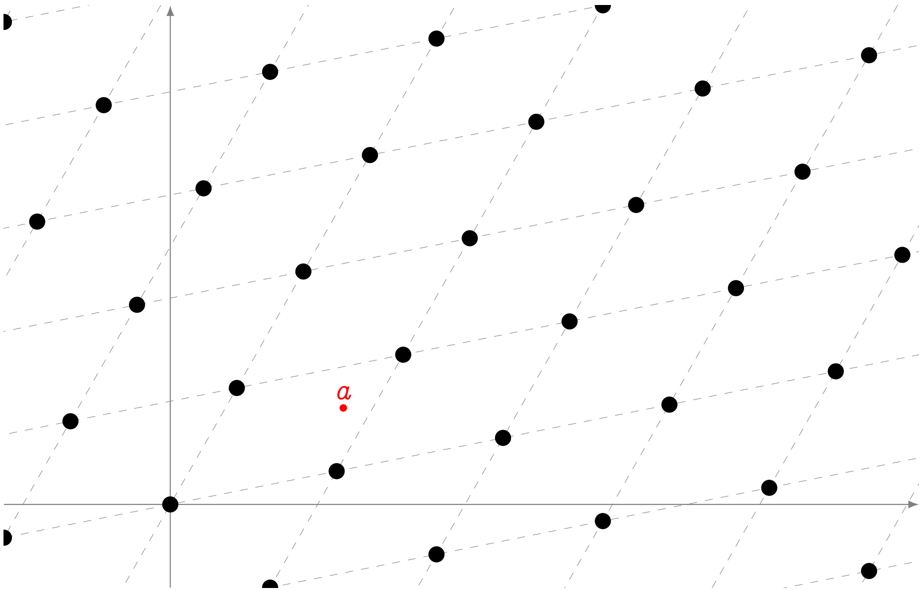
Multiplication



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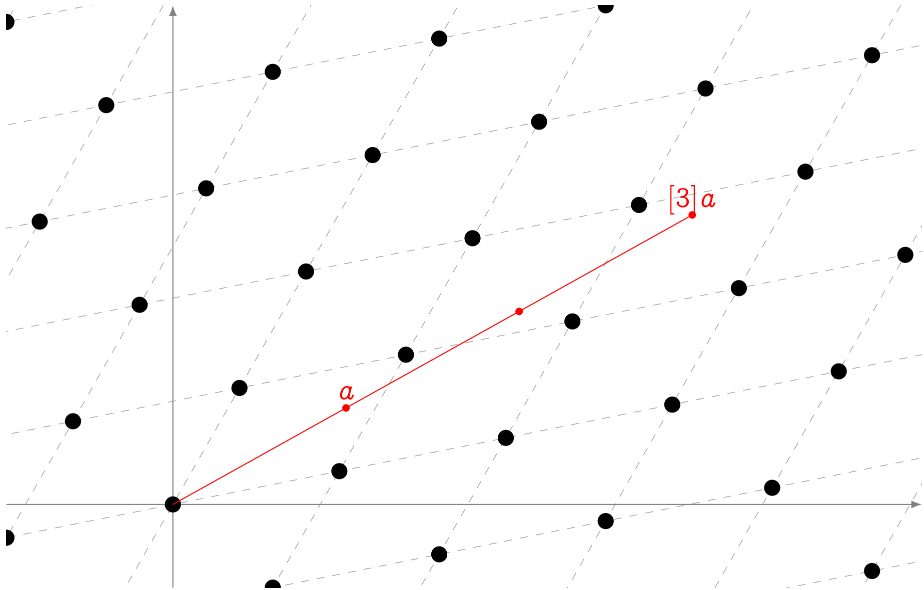

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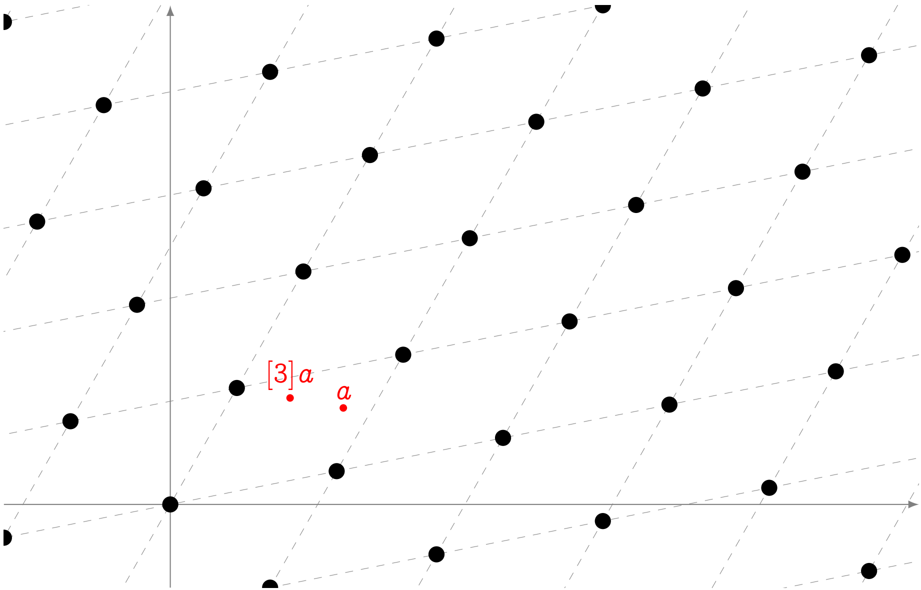
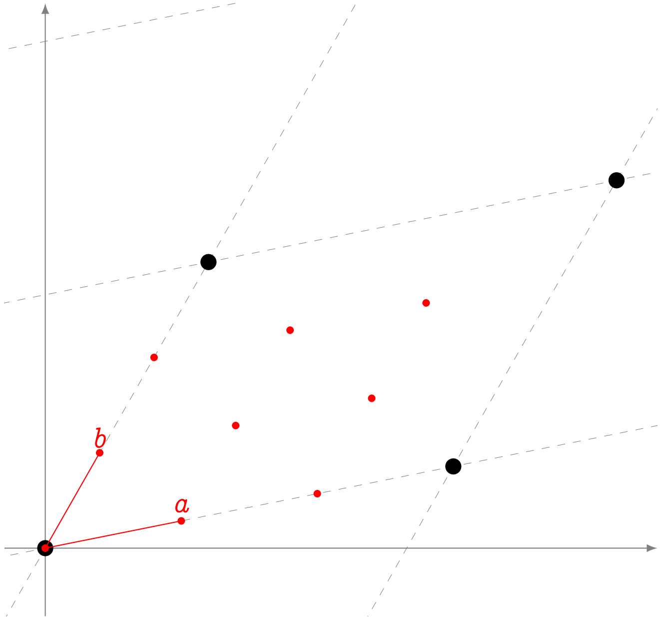


Torsion subgroups



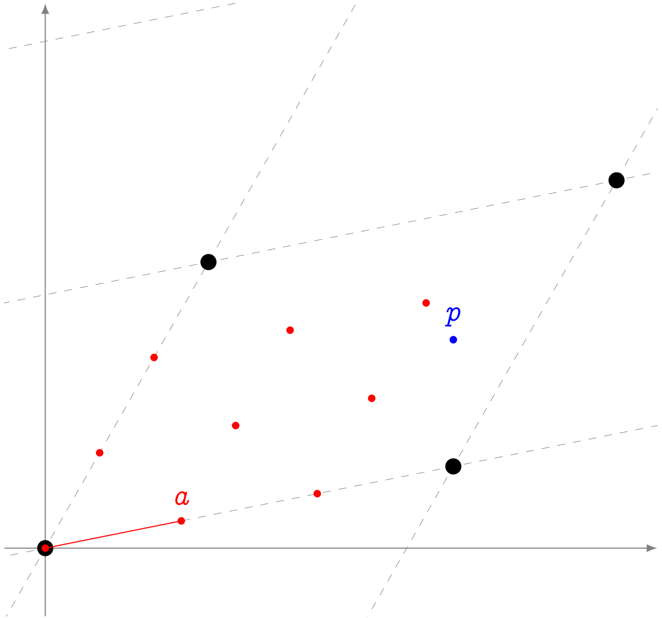
The ℓ-torsion subgroup is made up by the points

$$\left(\frac{i\omega_1}{\ell}, \frac{j\omega_2}{\ell}\right)$$

It is a group of rank two

$$E[\ell] = \langle a, b \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

Isogenies



Let $a \in \mathbb{C}/\Lambda_1$ be an ℓ-torsion point, and let

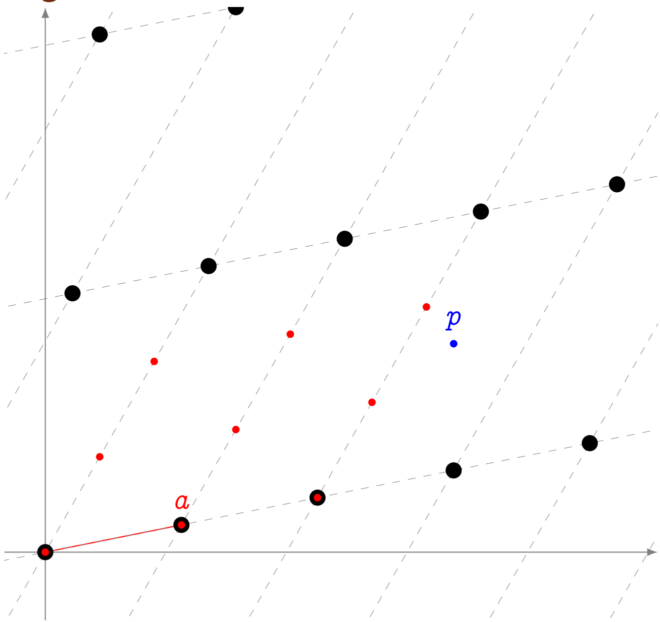
$$\Lambda_2 = a\mathbb{Z} \oplus \Lambda_1$$

Then $\Lambda_1 \subset \Lambda_2$ and we define a degree ℓ cover

$$\phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$$

ϕ is a morphism of complex Lie groups and is called an isogeny.

Isogenies



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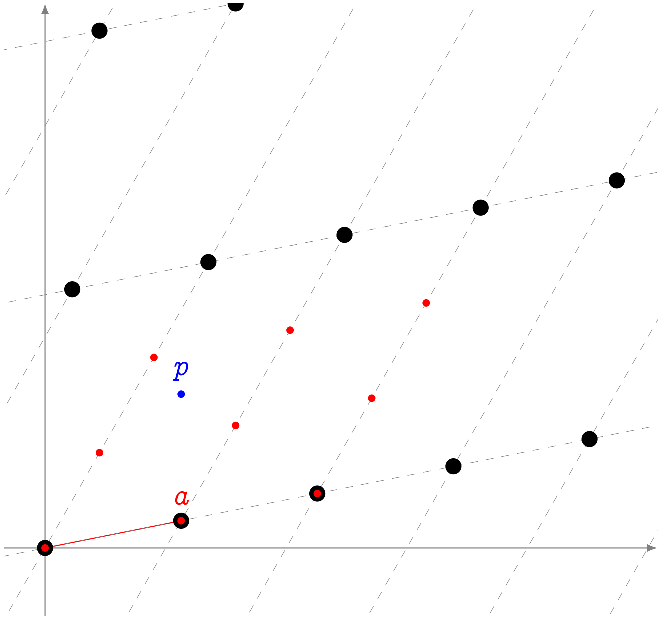

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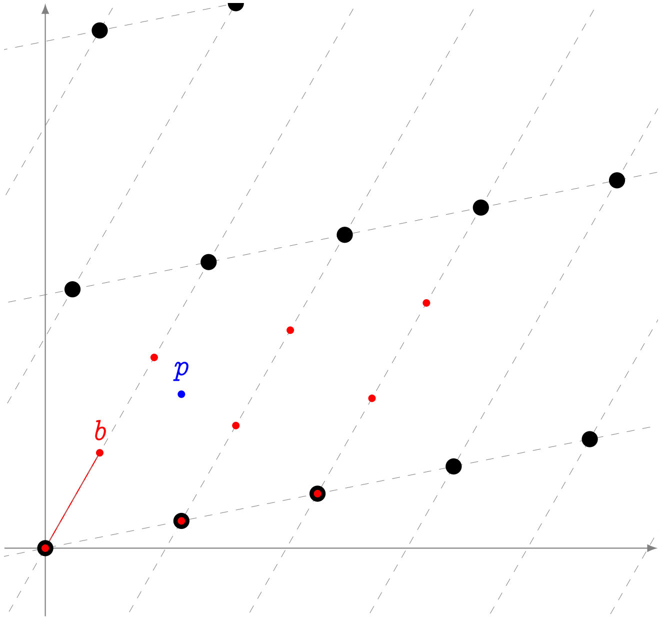


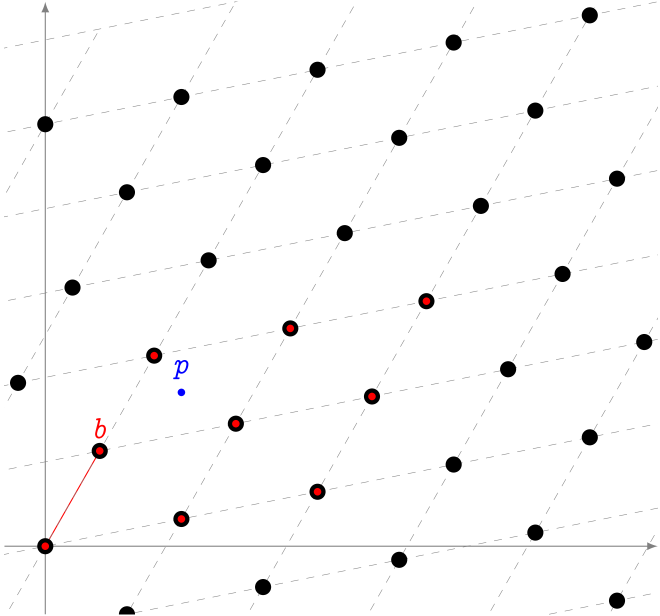
Taking a point b not in the kernel of ϕ, we obtain a new degree ℓ cover

$$\hat{\phi} : \mathbb{C}/\Lambda_2 \to \mathbb{C}/\Lambda_3$$

The composition $\hat{\phi} \circ \phi$ has degree ℓ^2 and is homothetic to the multiplication by ℓ map.

$\hat{\phi}$ is called the dual isogeny of ϕ.

Isogenies



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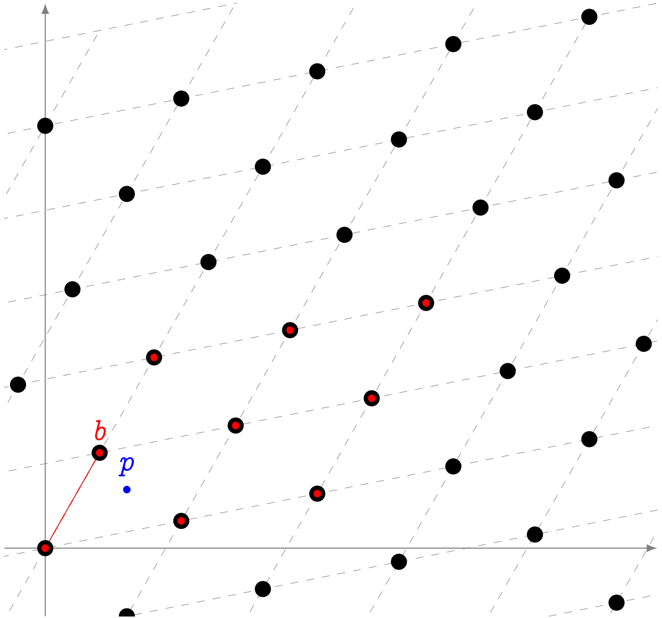

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Isogenies over arbitrary fields

Isogenies are just the right notion of morphism for elliptic curves

- Surjective group morphisms.
- Algebraic maps (i.e., defined by polynomials).

(Separable) isogenies ⇔ finite subgroups:

$$0 \to H \to E \xrightarrow{\phi} E' \to 0$$

The kernel H determines the image curve E' up to isomorphism

$$E/H \stackrel{\text{def}}{=} E'.$$

Isogeny degree

Neither of these definitions is quite correct, but they *nearly* are:

- The degree of ϕ is the cardinality of $\ker \phi$.
- (Bisson) the degree of ϕ is the time needed to compute it.

Easy and hard problems

In practice: an isogeny ϕ is just a rational fraction (or maybe two)

$$\frac{N(x)}{D(x)} = \frac{x^n + \cdots + n_1 x + n_0}{x^{n-1} + \cdots + d_1 x + d_0} \in k(x), \qquad \text{with } n = \deg \phi,$$

and $D(x)$ vanishes on $\ker \phi$.

The explicit isogeny problem

Input: A *description* of the isogeny (e.g, its kernel).

Output: The curve E/H and the rational fraction N/D.

Instances

- Input = kernel generator ▸ Velu's formulas; $\tilde{O}(n)$
- Input = E and E/H
  - ▸ Elkies' algorithm[a] (and variants); $\tilde{O}(n)$
  - ▸ Couveignes' algorithm[b] (and variants). $\tilde{O}(n^2)$

Lower bound: $\Omega(n)$.

[a] Elkies 1998.

[b] Couveignes 1996.

Easy and hard problems

The isogeny evaluation problem

Input: A *description* of the isogeny ϕ, a point $P \in E(k)$.

Output: The curve E/H and $\phi(P)$.

Examples

- Input = rational fraction; $O(n)$
- Input = composition of *low degree* isogenies; $\tilde{O}(\log n)$
- Input = kernel generator; $O(??)$
- Input = $\phi(a^{-1}P)$; $O(1)$

Easy and hard problems

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Exponential separation...

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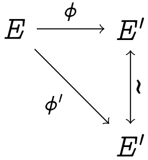
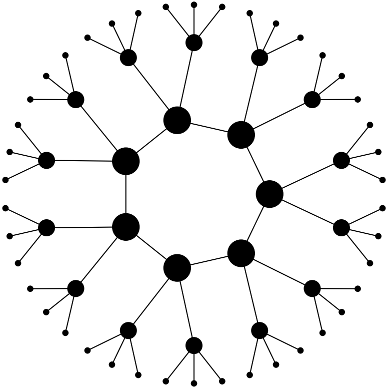
Exponential separation…Crypto happens!

Isogeny graphs

We look at the graph of elliptic curves with isogenies up to isomorphism. We say two isogenies ϕ, ϕ' are isomorphic if:



Example: Finite field, ordinary case, graph of isogenies of degree 3.



Structure of the graph[1]

Theorem (Serre-Tate)

Two curves are isogenous over a finite field k if and only if they have the same number of points on k.

The graph of isogenies of prime degree $\ell \neq p$

Ordinary case (isogeny volcanoes)

- Nodes can have degree $0, 1, 2$ or $\ell + 1$.
  - For $\sim 50\%$ of the primes ℓ, graphs are just isolated points;
  - For other $\sim 50\%$, graphs are 2-regular;
  - other cases only happen for finitely many ℓ's.

Supersingular case

- The graph is $\ell + 1$-regular.
- There is a unique (finite) connected component made of all supersingular curves with the same number of points.

[1] Deuring 1941; Kohel 1996; Fouquet and Morain 2002.

Expander graphs from isogenies

Expander graphs

An infinite family of connected k-regular graphs on n vertices is an expander family if there exists an $\epsilon > 0$ such that all non-trivial eigenvalues satisfy $|\lambda| \leq (1-\epsilon)k$ for n large enough.

- Expander graphs have short diameter ($O(\log n)$);
- Random walks mix rapidly (after $O(\log n)$ steps, the induced distribution on the vertices is close to uniform).

Supersingular Let ℓ be fixed, the graphs of all supersingular curves with ℓ-isogenies are expanders;[2]

Ordinary* Let $\mathcal{O} \subset \mathbb{Q}[\sqrt{-D}]$ be an order in a quadratic imaginary field. The graphs of all curves over $\mathbb{F}_q$ with complex multiplication by $\mathcal{O}$, with isogenies of prime degree bounded by $(\log q)^{2+\delta}$, are expanders.[3]

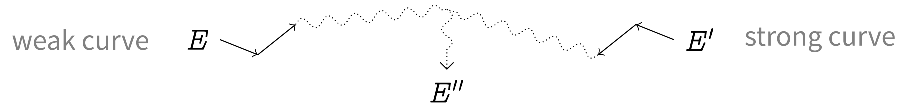
*(may contain traces of GRH)

[2] Pizer 1990, 1998.

[3] Jao, Miller, and Venkatesan 2009.

Isogeny walks and cryptanalysis[5]

(alternative) fact: Having a weak DLP is not (always) isogeny invariant.



Fourth root attacks

- Start two random walks from the two curves and wait for a collision.
- Over $\mathbb{F}_q$, the average size of an isogeny class is $h_\Delta \sim \sqrt{q}$.
- A collision is expected after $O(\sqrt{h_\Delta}) = O(q^{\frac{1}{4}})$ steps.

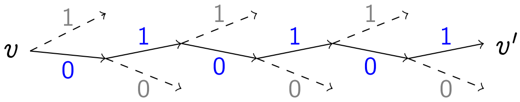
Note: Can be used to build trapdoor systems[4].

[4] Teske 2006.

[5] Steven D. Galbraith 1999; Steven D. Galbraith, Hess, and Smart 2002; Bisson and Sutherland 2011.

Random walks and hash functions

Any expander graph gives rise to a hash function.



$H(010101) = v'$

- Fix a starting vertex v;
- The value to be hashed determines a random path to v';
- v' is the hash.

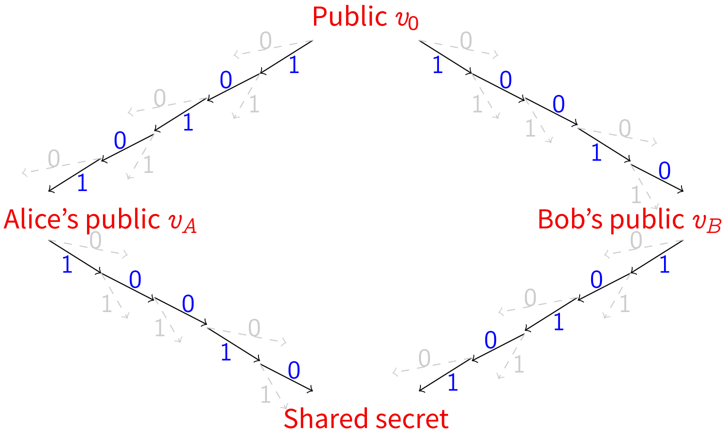
Provably secure hash functions

- Use the expander graph of supersingular 2-isogenies;[a]
- Collision resistance = hardness of finding cycles in the graph;
- Preimage resistance = hardness of finding a path from v to v'.

[a] Charles, K. E. Lauter, and Goren 2009.

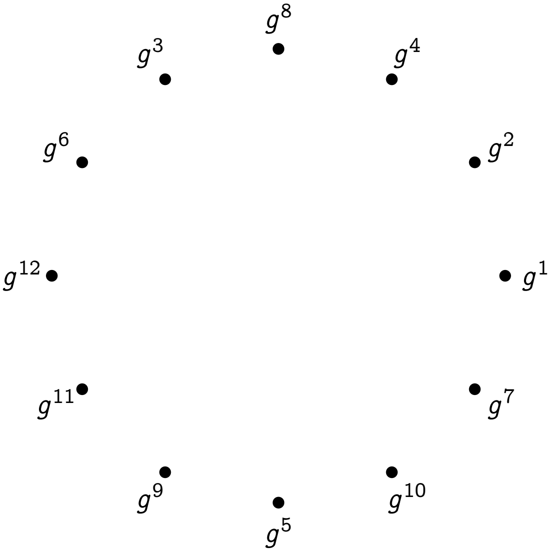
Random walks and key exchange

Let’s try something harder...



...is this even possible?

Expander graphs from groups



Let $G = \langle g \rangle$ be a cyclic group of order p.

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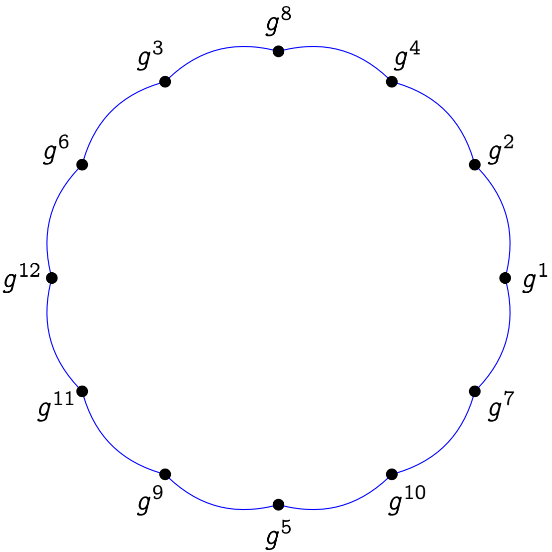

Let $G = \langle g \rangle$ be a cyclic group of order p. Let $S \subset (\mathbb{Z}/p\mathbb{Z})^\times$ s.t. $S^{-1} \subset S$.

The Schreier graph of $(S, G \setminus \{1\})$ is (usually) an expander.

— $x \mapsto x^2$

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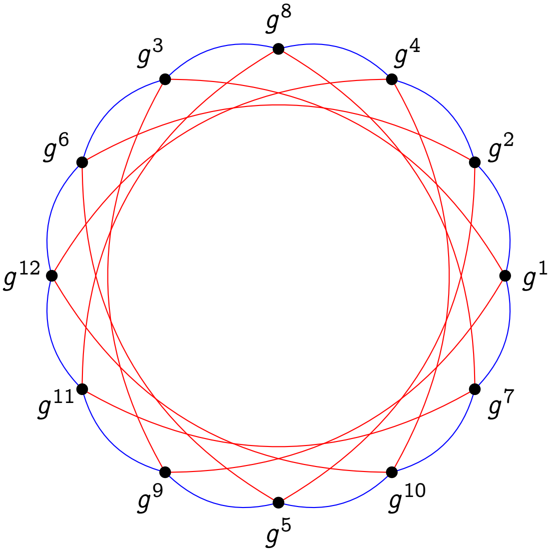

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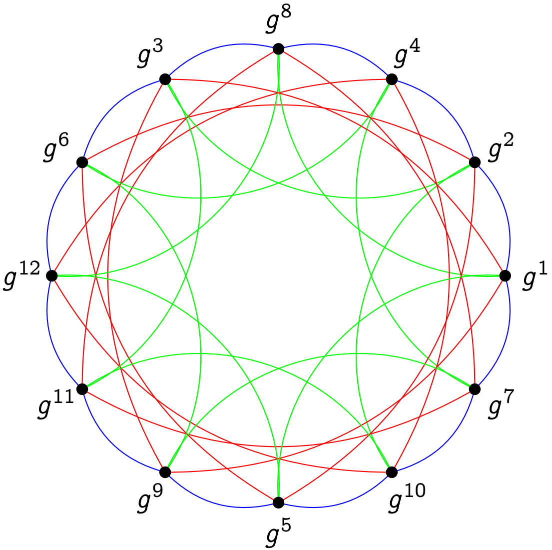

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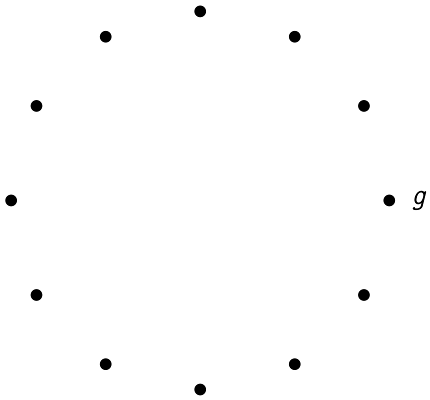
Key exchange from Schreier graphs



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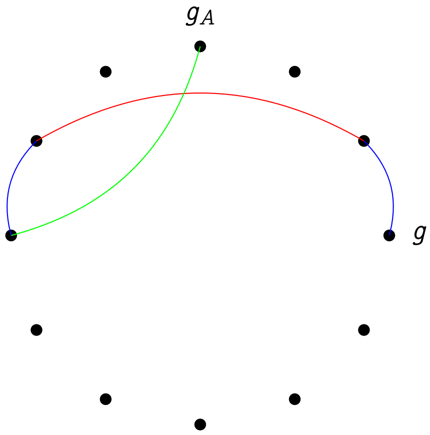

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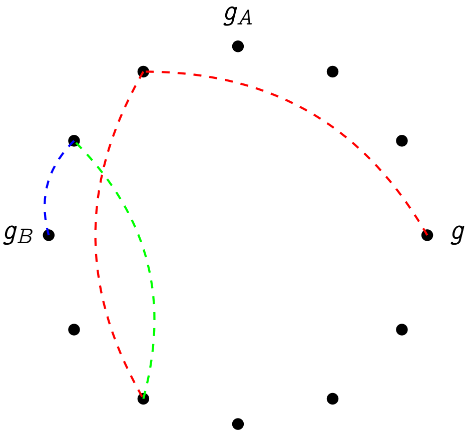

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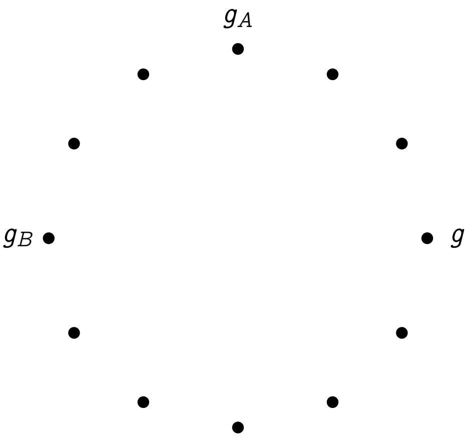

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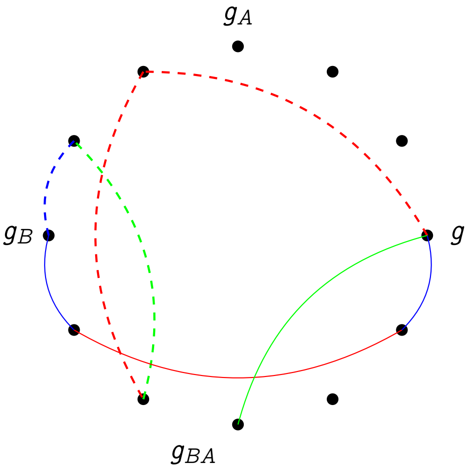

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4. **Alice** repeats her secret walk s_A starting from g_B.

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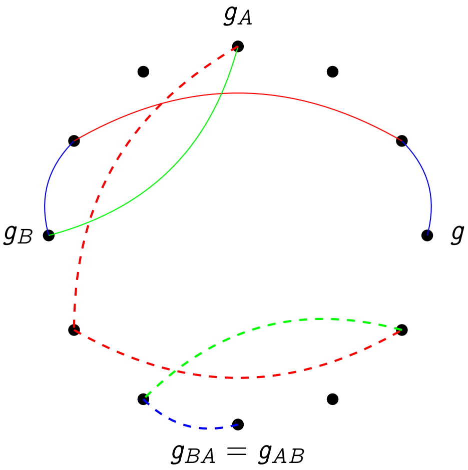

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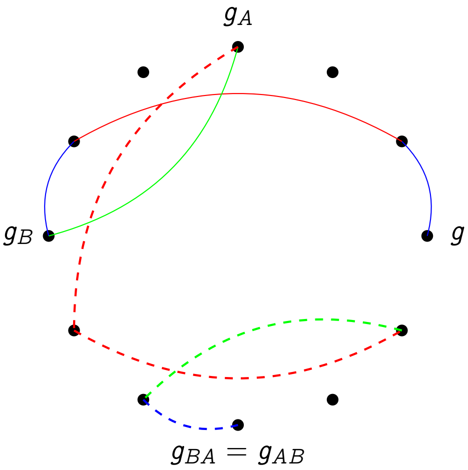


Why does this work?

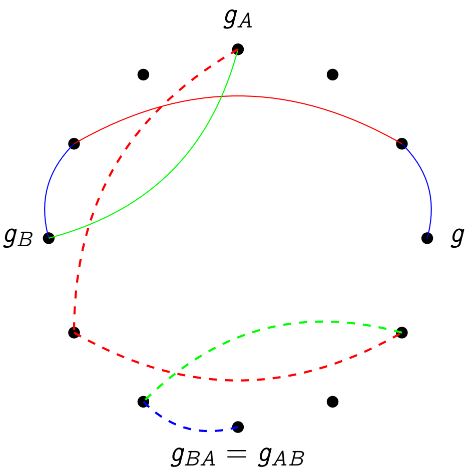
$$g_A = g^{2\cdot 3\cdot 2\cdot 5},$$

$$g_B = g^{3^2\cdot 5\cdot 2},$$

$$g_{BA} = g_{AB} = g^{2^3\cdot 3^3\cdot 5^2};$$

and g_A, g_B, g_{AB} are uniformly distributed in G…

Key exchange from Schreier graphs



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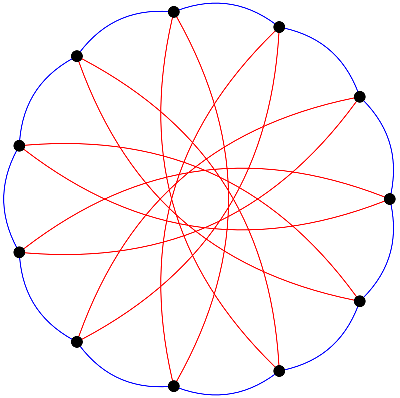
...Indeed, this is just a twisted presentation of the classical Diffie-Hellman protocol!

Group action on isogeny graphs



ℓ_1-isogenies

ℓ_2-isogenies

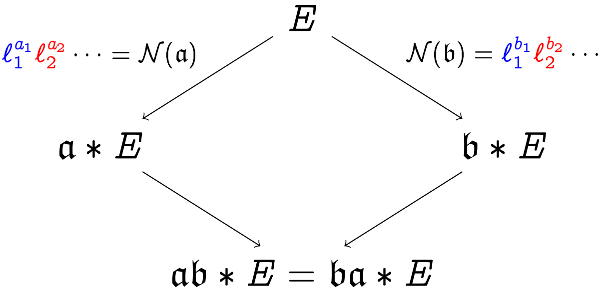
- There is a group action of the ideal class group $\mathrm{Cl}(\mathcal{O})$ on the set of ordinary curves with complex multiplication by $\mathcal{O}$.
- Its Schreier graph is an isogeny graph (and an expander if we take enough generators)

Key exchange in graphs of ordinary isogenies[6]

Parameters:

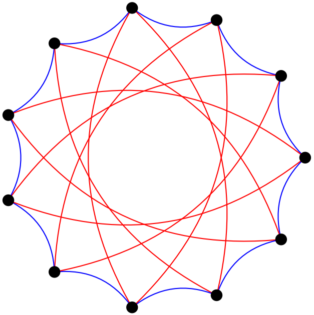
- $E/\mathbb{F}_p$ ordinary elliptic curve with Frobenius endomorphism π,
- primes ℓ_1,ℓ_2,...such that $\left(\frac{D_\pi}{\ell_i}\right) = 1$.
- A *direction* for each ℓ_i (i.e. an eigenvalue of π).

Secret data: Random walks $\mathfrak{a}, \mathfrak{b} \in \mathrm{Cl}(\mathcal{O})$ in the isogeny graph.



[6] Couveignes 2006; Rostovtsev and Stolbunov 2006.

R&S key exchange



Key generation: compose small degree isogenies
polynomial in the lenght of the random walk.

Attack: find an isogeny between two curves
polynomial in the degree, exponential in the length.

Quantum[7]: Shor + isogeny evaluation
subexponential in the length of the walk.

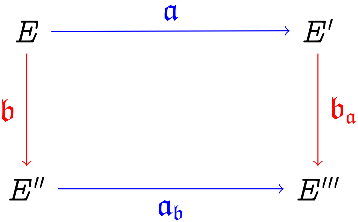
[7] Childs, Jao, and Soukharev 2010.

Key exchange with supersingular curves

Good news: there is no action of a commutative class group.

Bad news: there is no action of a commutative class group.

However: an algebraic structure is still acting on supersingular graphs: ideals of maximal orders of a quaternion algebra.



- The action is not commutative, we cannot use the same technique;
- We let instead Alice and Bob walk in two different isogeny graphs on the same vertex set.

Key exchange with supersingular curves

In practice, we fix:

- Small primes ℓ_A, ℓ_B;
- A large prime p such that $p + 1 = \ell_A^{e_A} \ell_B^{e_B}$;
- A supersingular curve E over $\mathbb{F}_{p^2}$, such that

$$E \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2 = (\mathbb{Z}/\ell_A^{e_A}\mathbb{Z})^2 \oplus (\mathbb{Z}/\ell_B^{e_B}\mathbb{Z})^2,$$

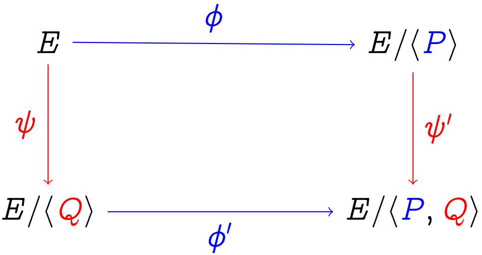
- We use isogenies of degrees $\ell_A^{e_A}$ and $\ell_B^{e_B}$ with cyclic rational kernels;
- The diagram below can be constructed in time poly($e_A + e_B$).

$$\ker \phi = \langle P \rangle \subset E[\ell_A^{e_A}]$$

$$\ker \psi = \langle Q \rangle \subset E[\ell_B^{e_B}]$$

$$\ker \phi' = \langle \psi(P) \rangle$$

$$\ker \psi' = \langle \phi(Q) \rangle$$



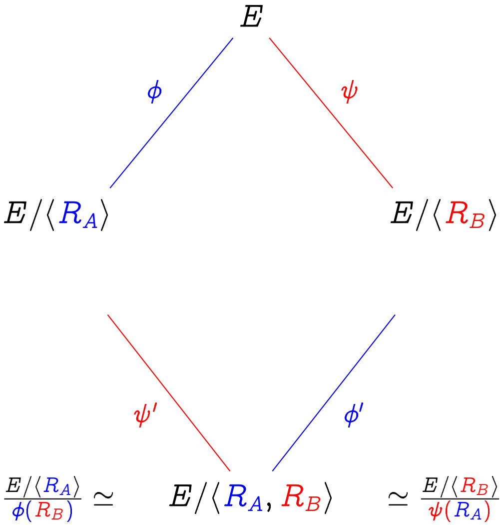
Supersingular Isogeny Diffie-Hellman[8]

Parameters:

- Prime p such that $p + 1 = \ell_A^a \ell_B^b$;
- Supersingular curve $E \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2$;
- $E[\ell_A^a] = \langle P_A, Q_A \rangle$;
- $E[\ell_B^b] = \langle P_B, Q_B \rangle$.

Secret data:

- $R_A = m_A P_A + n_A Q_A$,
- $R_B = m_B P_B + n_B Q_B$,



[8] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

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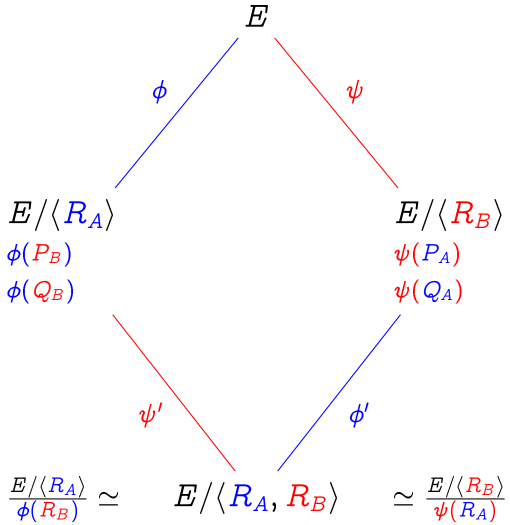

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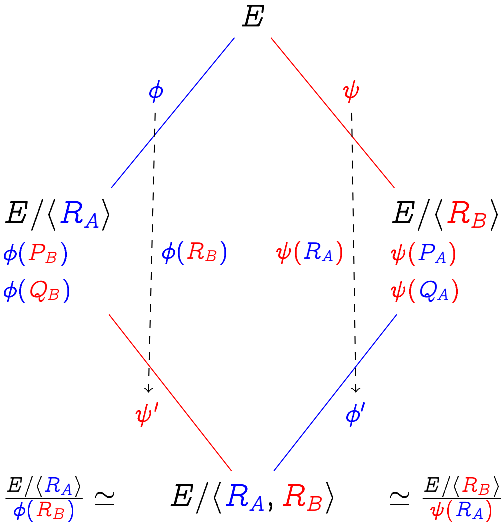


[8] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

Performance

- For efficiency choose p such that $p+1=2^a3^b$.
- For classical n-bit security, choose $2^a \sim 3^b \sim 2^{2n}$, hence $p \sim 2^{4n}$.
- For quantum n-bit security, choose $2^a \sim 3^b \sim 2^{3n}$, hence $p \sim 2^{6n}$.

Practical optimizations:

- Use new quasi-linear algorithm for isogeny evaluation[a].
- Optimize arithmetic for $\mathbb{F}_p$.[b,c]
- -1 is a quadratic non-residue: $\mathbb{F}_{p^2} \simeq \mathbb{F}_p[X]/(X^2+1)$.
- E (or its twist) has a 4-torsion point: use Montgomery form.
- Avoid inversions by using *projective curve equations*.[b]

Fastest implementation[b]: 100Mcycles (Intel Haswell) @128bits quantum security level, 4512bits public key size.

[a] De Feo, Jao, and Plût 2014.
[b] Costello, Longa, and Naehrig 2016.
[c] Karmakar, Roy, Vercauteren, and Verbauwhede 2016.

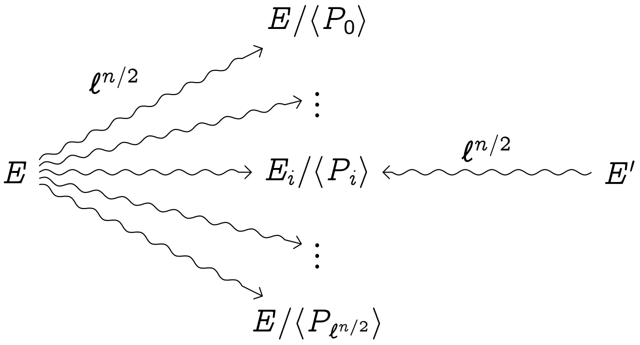
Comparison

	Speed	Communication
RSA 3072	4ms	0.3KiB
ECDH nistp256	0.7ms	0.03KiB
Code-based	0.5ms	360KiB
NTRU	0.3-1.2ms	1KiB
Ring-LWE	0.2-1.5ms	2-4KiB
LWE	1.4ms	11KiB
SIDH	35-400ms	0.5KiB

Source: D. Stebila, *Preparing for post-quantum cryptography in TLS*

Generic attacks

Problem: Given E, E', isogenous of degree ℓ^n, find $\phi : E \to E'$.



- With high probability ϕ is the unique collision (or *claw*).
- A quantum claw finding[9] algorithm solves the problem in $O(\ell^{n/3})$.

[9] Tani 2009.

Other attacks

Ephemeral key recovery (total break)

Given E_0 and a public curve $E_0/\langle R \rangle$, find the kernel of the secret isogeny:

Subexponential $L_p(1/2, \sqrt{3}/2)$ when both curves are defined over $\mathbb{F}_p$.[a]

Polynomial isomorphic problem on quaternion algebras.[b]

Equivalent to computing the endomorphism rings of both E_0 and $E_0/\langle R_A \rangle$.[c]

[a] Biasse, Jao, and Sankar 2014.
[b] Kohel, K. Lauter, Petit, and Tignol 2014.
[c] Steven D Galbraith, Petit, Shani, and Ti 2016.

Open problem: exploit the additional information transmitted by the protocol to improve attacks (classical or quantum).

Other attacks

Other security models

Active attack against long term keys, learns the full key with (close to) optimal number of oracle queries. Countermeasures are relatively expensive.[a]

Side channel Constant-time implementation available.[b]

Attack on partially leaked keys.[a,c]

[a] Steven D Galbraith, Petit, Shani, and Ti 2016.
[b] Costello, Longa, and Naehrig 2016.
[c] Two more papers at PQCrypto 2017

Open problem: Create a protocol secure against active adversaries.

Bonus: a ZK proof of knowledge[10]

Secret: knowledge of the kernel of a degree $\ell_A^{e_A}$ isogeny from E to $E/\langle S\rangle$.

$$E \xrightarrow{\quad \phi \quad} E/\langle S\rangle$$

[10] De Feo, Jao, and Plût 2014.

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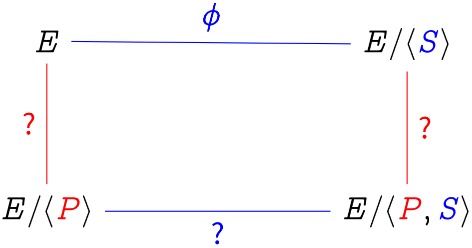

1. Choose a random point $P \in E[\ell_B^{e_B}]$, compute the diagram;
2. Publish the curves $E/\langle P\rangle$ and $E/\langle P, S\rangle$;

[10] De Feo, Jao, and Plût 2014.

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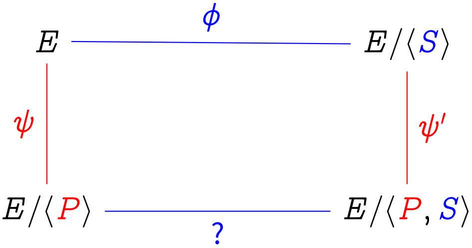

1. Choose a random point $P \in E[\ell_B^{e_B}]$, compute the diagram;
2. Publish the curves $E/\langle P\rangle$ and $E/\langle P, S\rangle$;
3. The verifier asks one of the two questions:
   - Reveal the degree $\ell_B^{e_B}$ isogenies;

[10] De Feo, Jao, and Plût 2014.

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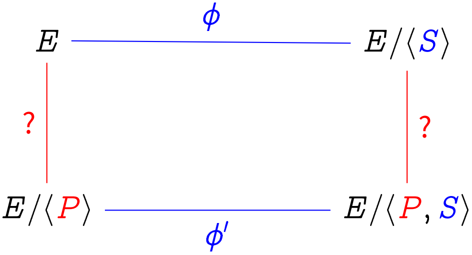

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2. Publish the curves $E/\langle P\rangle$ and $E/\langle P, S\rangle$;
3. The verifier asks one of the two questions:
    - Reveal the degree $\ell_B^{e_B}$ isogenies;
    - Reveal the bottom isogeny.

[10] De Feo, Jao, and Plût 2014.

Other protocols based on SIDH

Non-interactive protocols

- El-Gamal encryption.

Interactive protocols

- Signatures (using Fiat-Shamir)[a],
- Undeniable signatures[b],
- Strong designated verifier signatures[c],
- Authenticated encryption[d].

---

[a] Steven D Galbraith, Petit, Shani, and Ti 2016.
[b] Jao and Soukharev 2014.
[c] Sun, Tian, and Wang 2012.
[d] Soukharev, Jao, and Seshadri 2016.

Open problem: Classical signatures, …

Thank you

http://defeo.lu/

@luca_defeo

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