# Open problems in isogeny-based cryptography


**Luca De Feo**

**Université de Versailles & Inria, Université Paris-Saclay**


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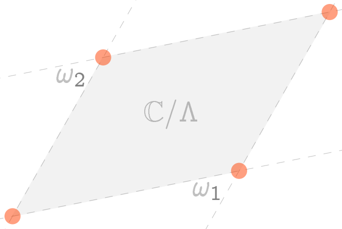
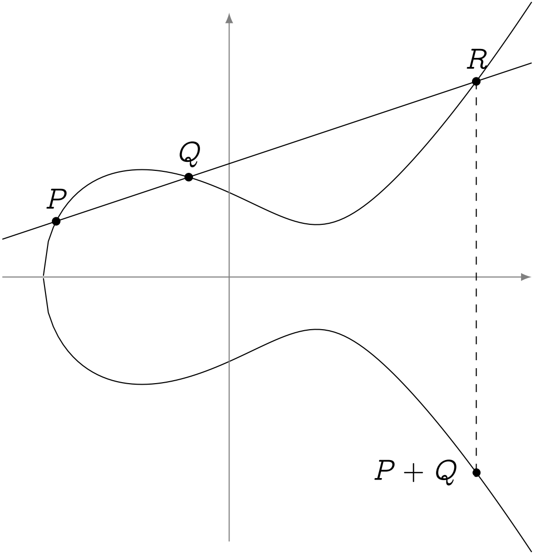


# Overview

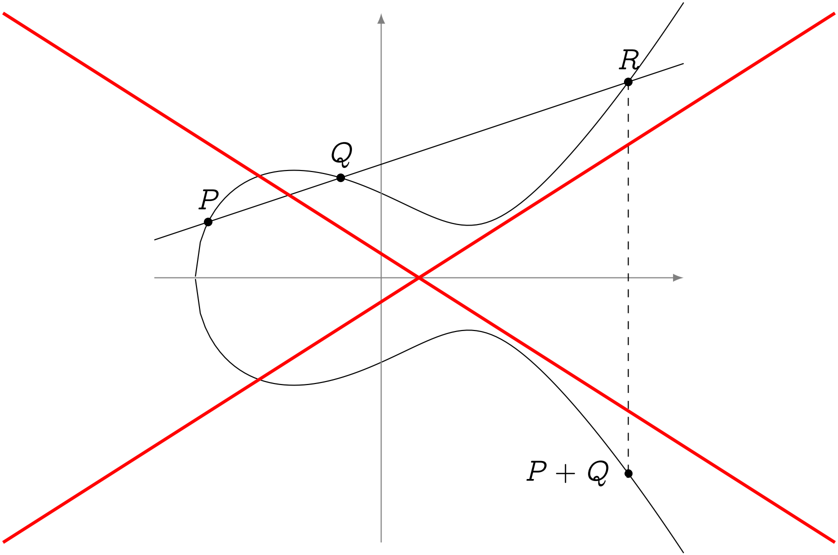
# Elliptic curves

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve…



# Elliptic curves

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve…forget it!



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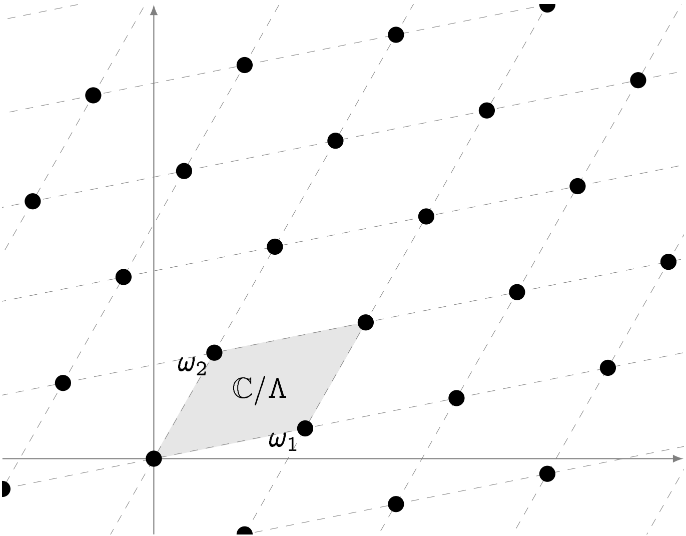
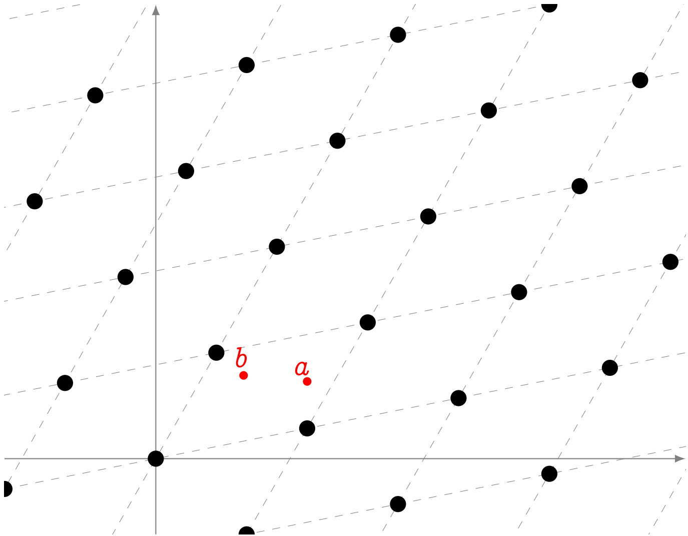



Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent complex numbers. Set

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$$

$\mathbb{C}/\Lambda$ is an elliptic curve.

# Elliptic curves




Addition law induced by addition on $\mathbb{C}$.

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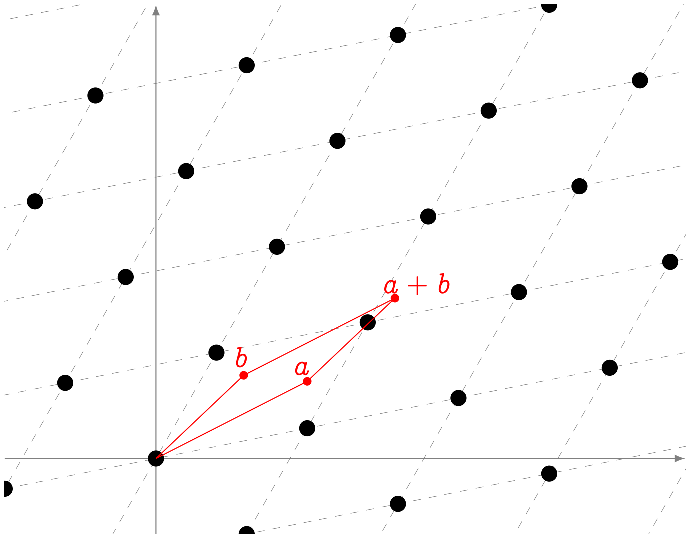


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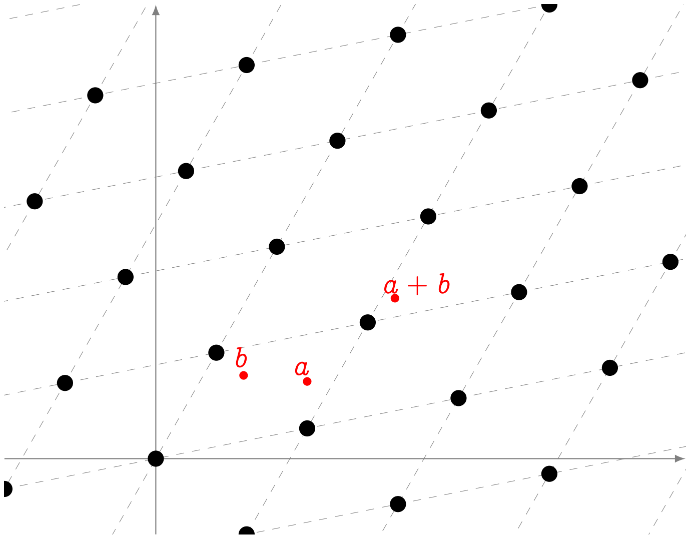


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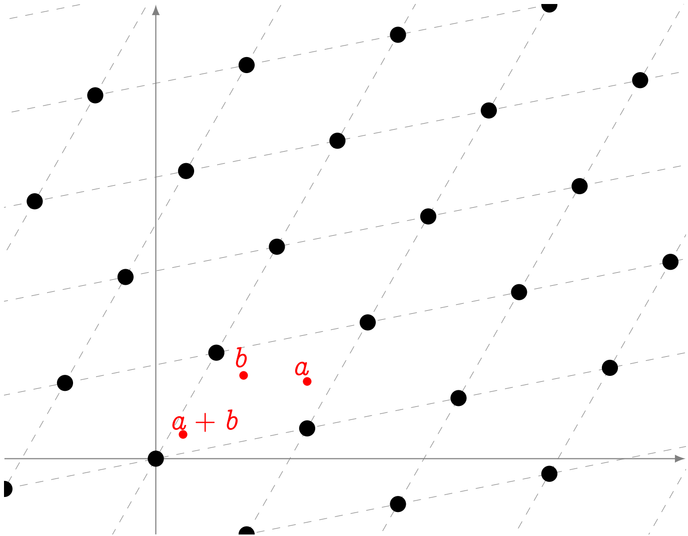


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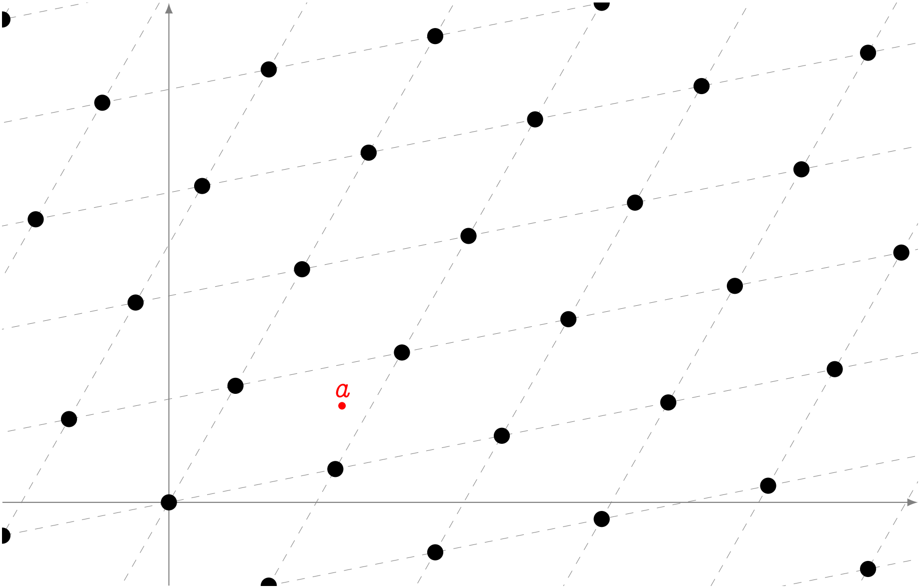
# Multiplication



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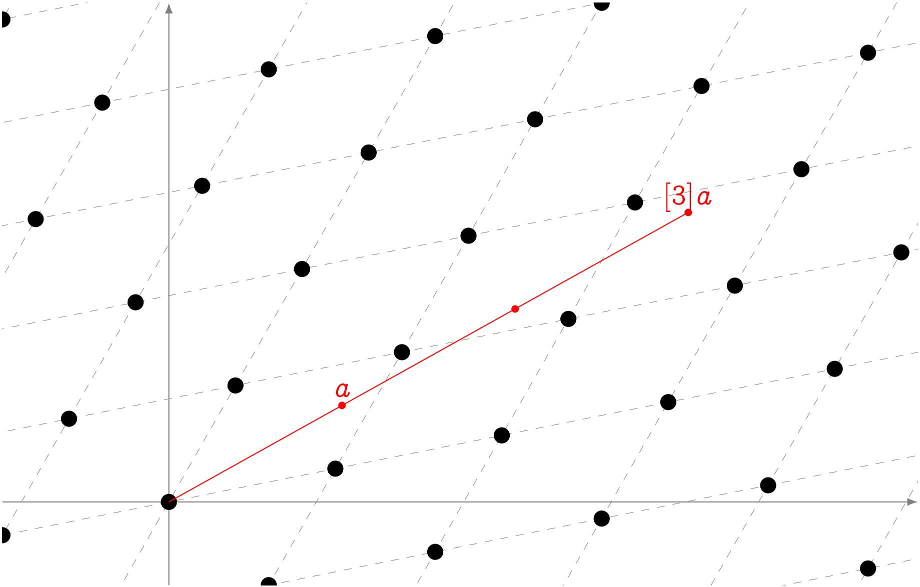

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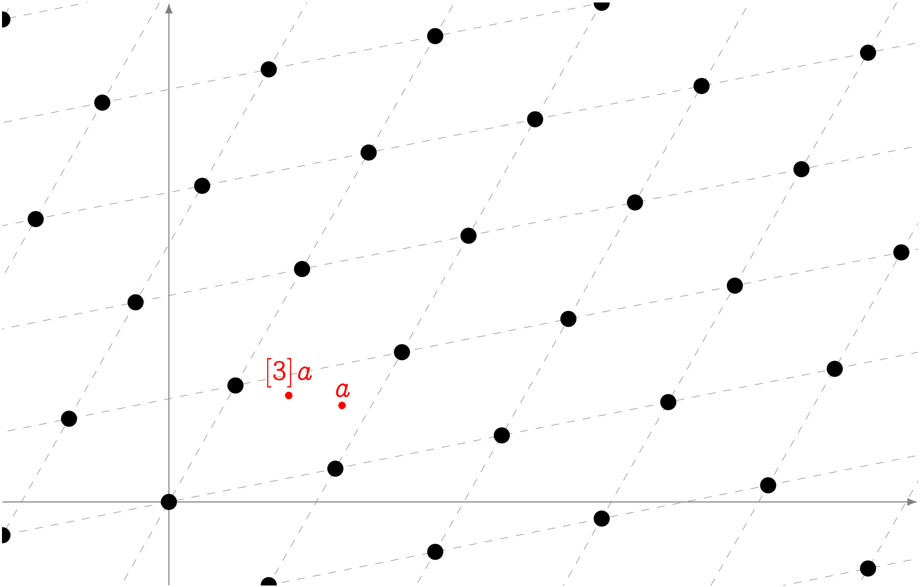
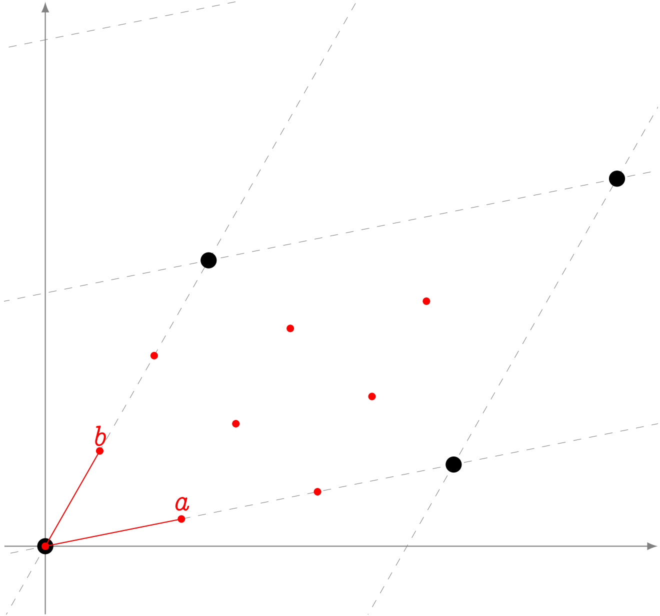


# Torsion subgroups



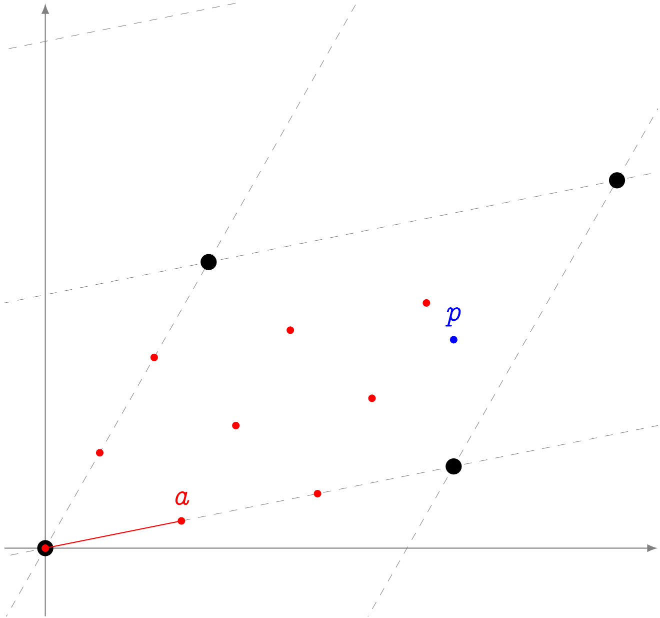

The $\ell$-torsion subgroup is made up by the points

$$\left(\frac{i\omega_1}{\ell}, \frac{j\omega_2}{\ell}\right)$$

It is a group of rank two

$$E[\ell] = \langle a, b \rangle \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$$

# Isogenies




Let $a \in \mathbb{C}/\Lambda_1$ be an $\ell$-torsion point, and let

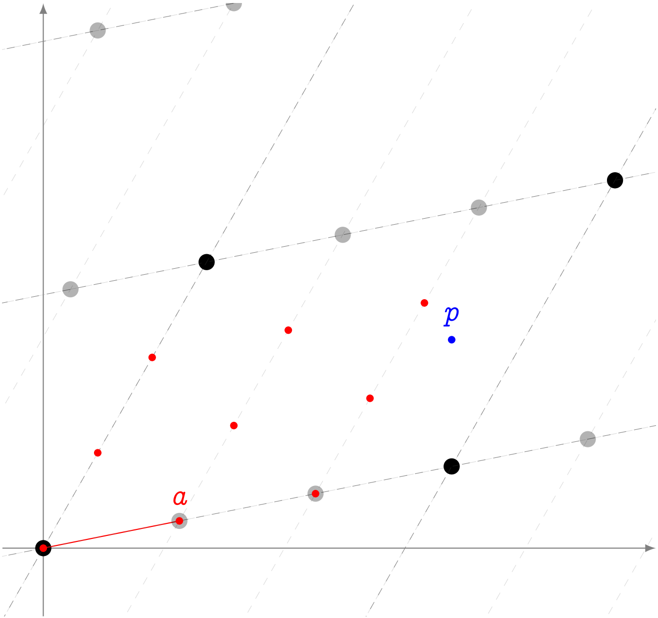
$$\Lambda_2 = a\mathbb{Z} \oplus \Lambda_1$$

Then $\Lambda_1 \subset \Lambda_2$ and we define a degree $\ell$ cover

$$\phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$$

$\phi$ is a morphism of complex Lie groups and is called an isogeny.

# Isogenies

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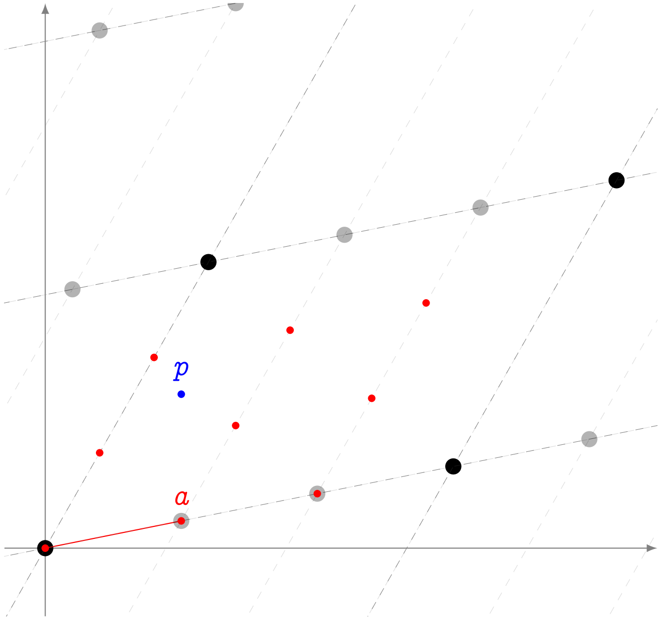


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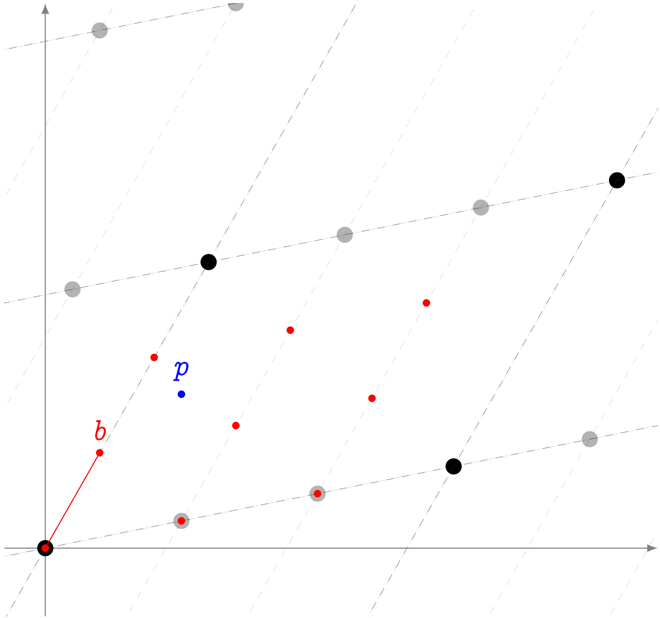


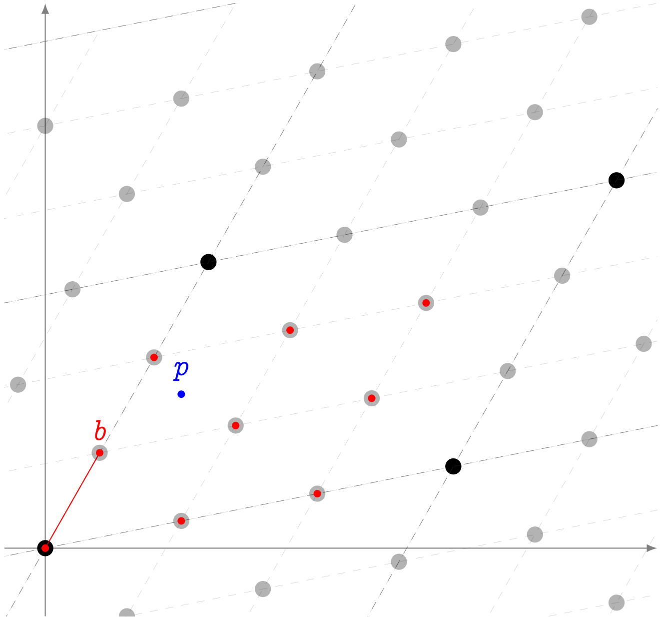

Taking a point $b$ not in the kernel of $\phi$, we obtain a new degree $\ell$ cover

$$\hat{\phi} : \mathbb{C}/\Lambda_2 \to \mathbb{C}/\Lambda_3$$

The composition $\hat{\phi} \circ \phi$ has degree $\ell^2$ and is homothetic to the multiplication by $\ell$ map.

$\hat{\phi}$ is called the dual isogeny of $\phi$.

# Isogenies

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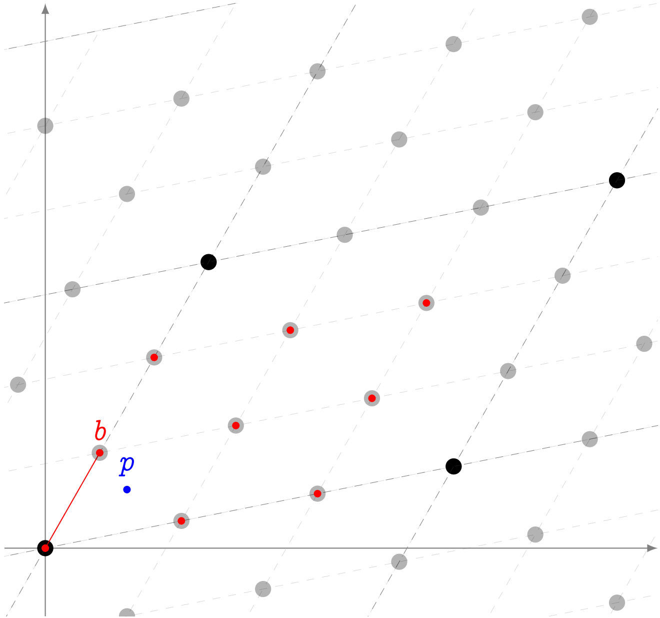


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# Isogenies over arbitrary fields

Isogenies are just the right notion of morphism for elliptic curves

- Surjective group morphisms.
- Algebraic maps (i.e., defined by polynomials).

(Separable) isogenies $\Leftrightarrow$ finite subgroups:

$$0 \to H \to E \xrightarrow{\phi} E' \to 0$$

The kernel $H$ determines the image curve $E'$ up to isomorphism

$$E/H \overset{\text{def}}{=} E'.$$

## Isogeny degree

Neither of these definitions is quite correct, but they *nearly* are:

- The degree of $\phi$ is the cardinality of $\ker \phi$.
- (Bisson) the degree of $\phi$ is the time needed to compute it.

# Easy and hard problems

In practice: an isogeny $\phi$ is just a rational fraction (or maybe two)

$$\frac{N(x)}{D(x)} = \frac{x^n + \cdots + n_1 x + n_0}{x^{n-1} + \cdots + d_1 x + d_0} \in k(x), \qquad \text{with } n = \deg \phi,$$

and $D(x)$ vanishes on $\ker \phi$.

## The explicit isogeny problem

Input: A *description* of the isogeny (e.g, its kernel).

Output: The curve $E/H$ and the rational fraction $N/D$.

Instances
- Input = kernel generator ▸ Velu's formulas; $\tilde{O}(n)$
- Input = $E$ and $E/H$
  - Elkies' algorithm[a] (and variants); $\tilde{O}(n)$
  - Couveignes' algorithm[b] (and variants). $\tilde{O}(n^2)$

Lower bound: $\Omega(n)$.

[a] Elkies 1998.

[b] Couveignes 1996.

# Easy and hard problems

## The isogeny evaluation problem

Input: A *description* of the isogeny $\phi$, a point $P \in E(k)$.

Output: The curve $E/H$ and $\phi(P)$.

Examples
- Input = rational fraction; $O(n)$
- Input = composition of *low degree* isogenies; $\tilde{O}(\log n)$
- Input = kernel generator; $O(??)$

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**Exponential separation...**

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- Input = rational fraction; $O(n)$
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- Input = kernel generator; $O(??)$

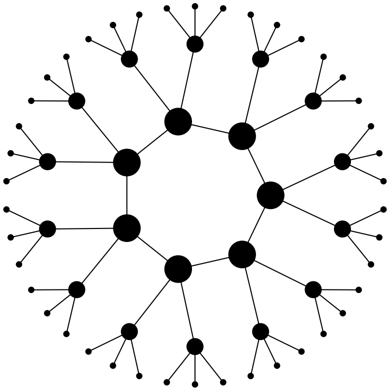
**Exponential separation...Crypto happens!**

# Isogeny graphs

We look at the graph of elliptic curves with isogenies up to isomorphism. We say two isogenies $\phi, \phi'$ are isomorphic if:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & E' \arrow[d, leftrightarrow, "\wr"] \\ & E' \end{tikzcd}$$

Example: Finite field, ordinary case, graph of isogenies of degree 3.



# Structure of the graph[1]

## Theorem (Serre-Tate)

*Two curves are isogenous over a finite field $k$ if and only if they have the same number of points on $k$.*

## The graph of isogenies of prime degree $\ell \neq p$

### Ordinary case (isogeny volcanoes)

- Nodes can have degree $0, 1, 2$ or $\ell + 1$.
  - For $\sim 50\%$ of the primes $\ell$, graphs are just isolated points;
  - For other $\sim 50\%$, graphs are 2-regular;
  - other cases only happen for finitely many $\ell$'s.

### Supersingular case

- The graph is $\ell + 1$-regular.
- There is a unique (finite) connected component made of all supersingular curves with the same number of points.

[1] Deuring 1941; Kohel 1996; Fouquet and Morain 2002.

# Expander graphs from isogenies

## Expander graphs

An infinite family of connected $k$-regular graphs on $n$ vertices is an expander family if there exists an $\epsilon > 0$ such that all non-trivial eigenvalues satisfy $|\lambda| \leq (1-\epsilon)k$ for $n$ large enough.

- Expander graphs have short diameter ($O(\log n)$);
- Random walks mix rapidly (after $O(\log n)$ steps, the induced distribution on the vertices is close to uniform).

**Supersingular** Let $\ell$ be fixed, the graphs of all supersingular curves with $\ell$-isogenies are expanders;[2]

**Ordinary*** Let $\mathcal{O} \subset \mathbb{Q}[\sqrt{-D}]$ be an order in a quadratic imaginary field. The graphs of all curves over $\mathbb{F}_q$ with complex multiplication by $\mathcal{O}$, with isogenies of prime degree bounded by $(\log q)^{2+\delta}$, are expanders.[3]

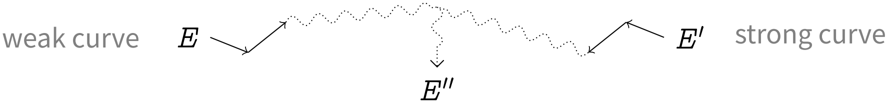
*(may contain traces of GRH)

[2] Pizer 1990, 1998.

[3] Jao, Miller, and Venkatesan 2009.

# Isogeny walks and cryptanalysis[5]

(alternative) fact: Having a weak DLP is not (always) isogeny invariant.




## Fourth root attacks

- Start two random walks from the two curves and wait for a collision.
- Over $\mathbb{F}_q$, the average size of an isogeny class is $h_\Delta \sim \sqrt{q}$.
- A collision is expected after $O(\sqrt{h_\Delta}) = O(q^{\frac{1}{4}})$ steps.

Note: Can be used to build trapdoor systems[4].

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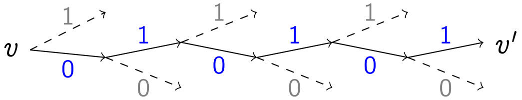
[4] Teske 2006.

[5] Steven D. Galbraith 1999; Steven D. Galbraith, Hess, and Smart 2002; Bisson and Sutherland 2011.

# Random walks and hash functions

Any expander graph gives rise to a hash function.




$$H(010101) = v'$$

- Fix a starting vertex $v$;
- The value to be hashed determines a random path to $v'$;
- $v'$ is the hash.

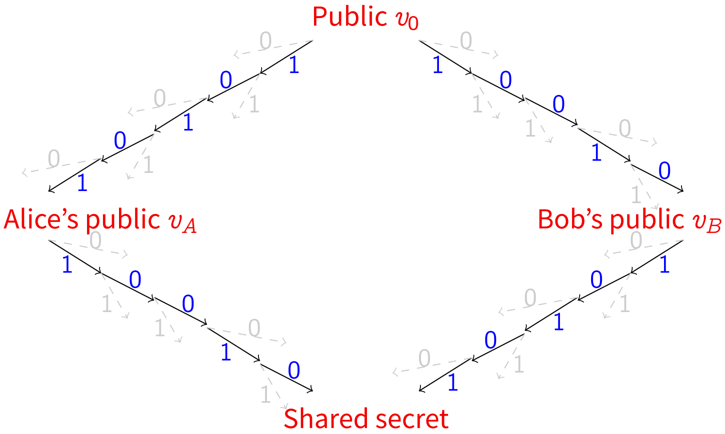
## Provably secure hash functions

- Use the expander graph of supersingular 2-isogenies;[a]
- Collision resistance = hardness of finding cycles in the graph;
- Preimage resistance = hardness of finding a path from $v$ to $v'$.
- Partly broken, known weak instances.[b]

[a] Charles, K. E. Lauter, and Goren 2009.
[b] Kohel, K. Lauter, Petit, and Tignol 2014.

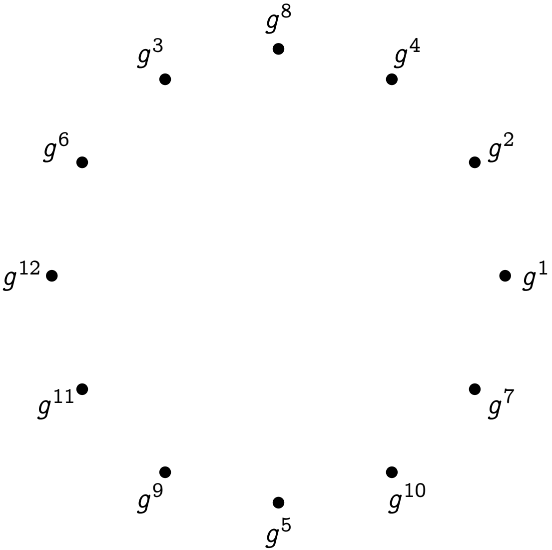
# Random walks and key exchange

Let's try something harder...




...is this even possible?

# Expander graphs from groups




Let $G = \langle g \rangle$ be a cyclic group of order $p$.

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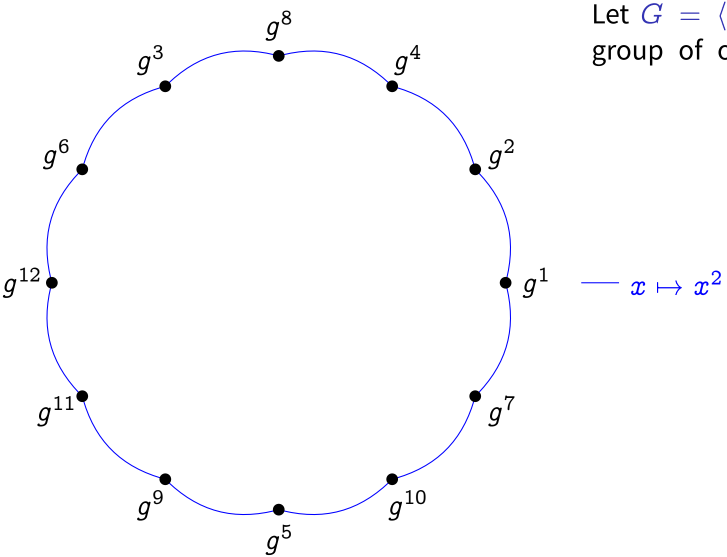


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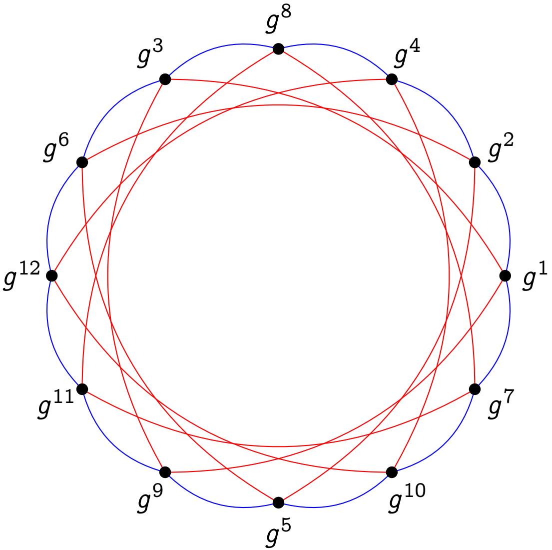


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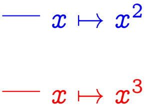

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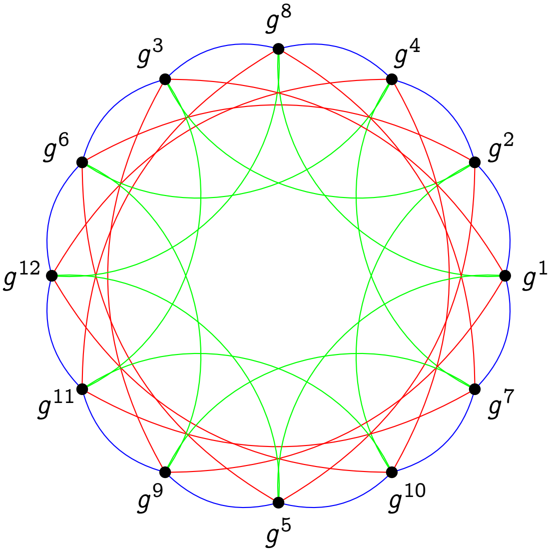


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— $x \mapsto x^2$

— $x \mapsto x^3$

— $x \mapsto x^5$

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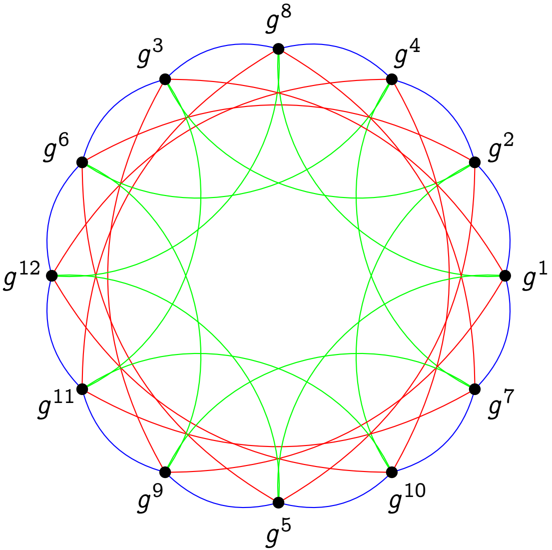


Let $G = \langle g \rangle$ be a cyclic group of order $p$. Let $S \subset (\mathbb{Z}/p\mathbb{Z})^\times$ s.t. $S^{-1} \subset S$.

The Schreier graph of $(S, G \setminus \{1\})$ is (usually) an expander.

— $x \mapsto x^2$

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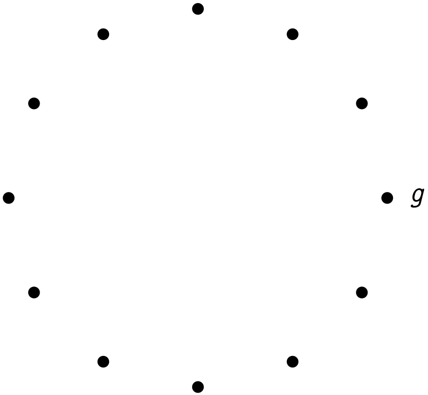
— $x \mapsto x^5$

# Key exchange from Schreier graphs




**Public parameters:**

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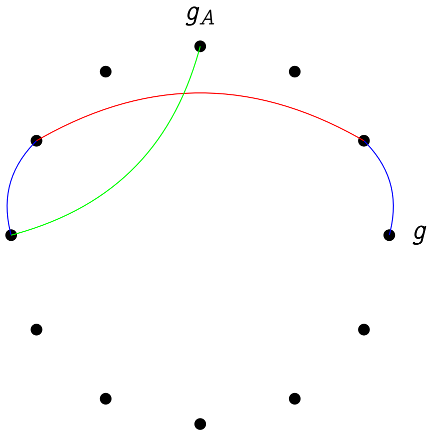


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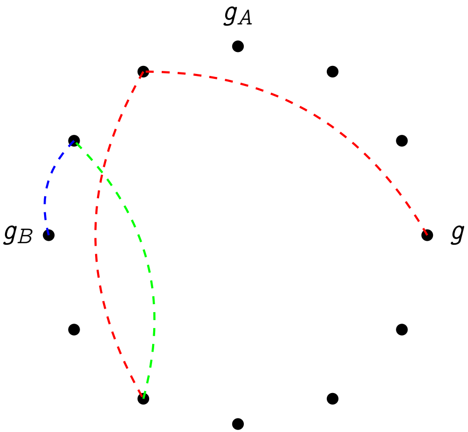


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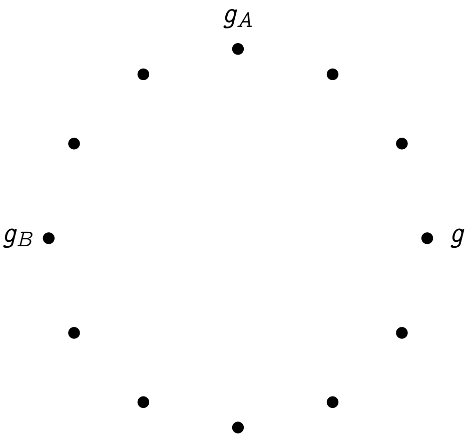


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3. They publish $g_A$ and $g_B$;

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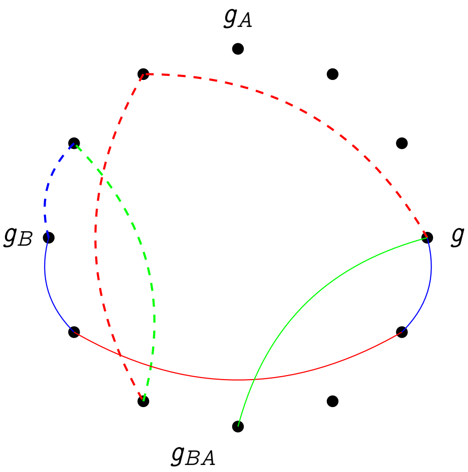


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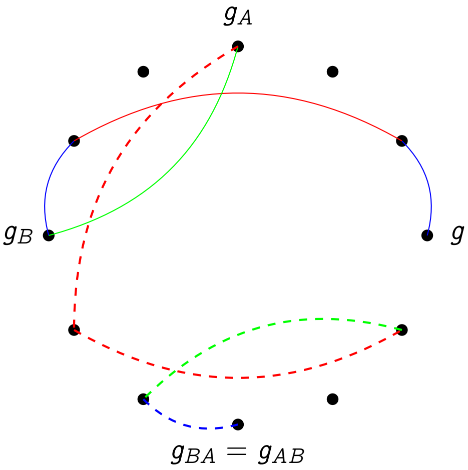


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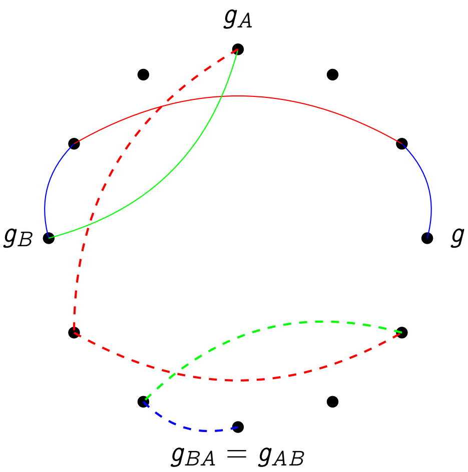


**Why does this work?**

$$g_A = g^{2\cdot 3\cdot 2\cdot 5},$$

$$g_B = g^{3^2\cdot 5\cdot 2},$$

$$g_{BA} = g_{AB} = g^{2^3\cdot 3^3\cdot 5^2};$$

and $g_A$, $g_B$, $g_{AB}$ are uniformly distributed in $G$…

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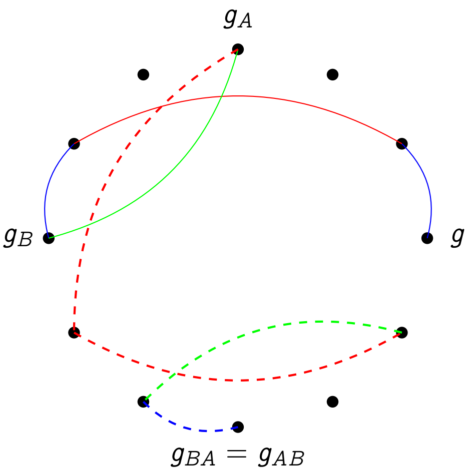


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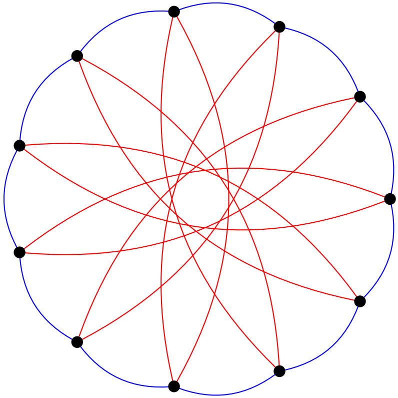
and $g_A$, $g_B$, $g_{AB}$ are uniformly distributed in $G$…

…Indeed, this is just a twisted presentation of the classical Diffie-Hellman protocol!

# Group action on isogeny graphs



— $\ell_1$-isogenies

— $\ell_2$-isogenies

- There is a group action of the ideal class group $\mathrm{Cl}(\mathcal{O})$ on the set of ordinary curves with complex multiplication by $\mathcal{O}$.
- Its Schreier graph is an isogeny graph (and an expander if we take enough generators)

# Key exchange in graphs of ordinary isogenies[6]

Parameters:

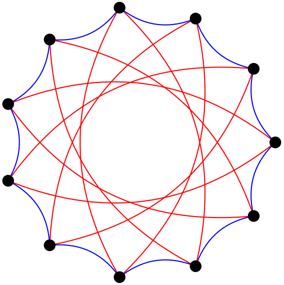
- $E/\mathbb{F}_p$ ordinary elliptic curve with *Frobenius endomorphism* $\pi$,
- primes $\ell_1$,$\ell_2$,... such that $\left(\frac{D_\pi}{\ell_i}\right) = 1$.
- A *direction* for each $\ell_i$ (i.e. an eigenvalue of $\pi$).

Secret data: Random walks $\mathfrak{a}, \mathfrak{b} \in \mathrm{Cl}(\mathcal{O})$ in the isogeny graph.

$$\begin{tikzcd}
 & E \arrow[dl, "\ell_1^{a_1}\ell_2^{a_2}\cdots = \mathcal{N}(\mathfrak{a})"'] \arrow[dr, "\mathcal{N}(\mathfrak{b}) = \ell_1^{b_1}\ell_2^{b_2}\cdots"] & \\
\mathfrak{a} * E \arrow[dr] & & \mathfrak{b} * E \arrow[dl] \\
 & \mathfrak{ab} * E = \mathfrak{ba} * E &
\end{tikzcd}$$

[6] Couveignes 2006; Rostovtsev and Stolbunov 2006.

# R&S key exchange



Key generation: compose small degree isogenies
polynomial in the length of the random walk.

Attack: find an isogeny between two curves
polynomial in the degree, exponential in the length.

Quantum[7]: QFT (hidden shift problem) + isogeny evaluation
subexponential in the length of the walk.

Open problem: Make this thing practical!

[7] Childs, Jao, and Soukharev 2010.

# Key exchange with supersingular curves

Good news: there is no action of a commutative class group.

Bad news: there is no action of a commutative class group.

Idea: Let Alice and Bob walk in two different isogeny graphs on the same vertex set.

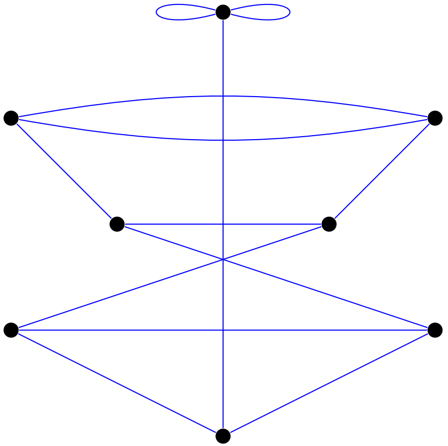


Figure: 2- and 3-isogeny graphs on $\mathbb{F}_{97^2}$.

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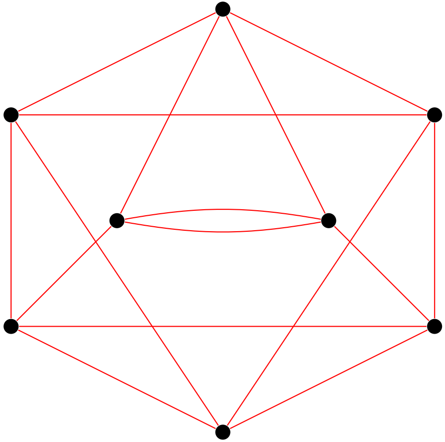

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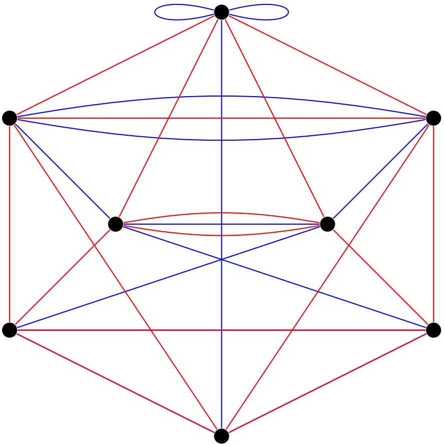

Figure: 2- and 3-isogeny graphs on $\mathbb{F}_{97^2}$.

# Key exchange with supersingular curves

- Fix small primes $\ell_A$, $\ell_B$;
- No canonical labeling of the $\ell_A$- and $\ell_B$-isogeny graphs; however…

**Walk of length $e_A$**

$$=$$

**Isogeny of degree $\ell_A^{e_A}$**

$$=$$

**Kernel $\langle P\rangle \subset E[\ell_A^{e_A}]$**

$$\ker\phi = \langle P\rangle \subset E[\ell_A^{e_A}]$$
$$\ker\psi = \langle Q\rangle \subset E[\ell_B^{e_B}]$$
$$\ker\phi' = \langle \psi(P)\rangle$$
$$\ker\psi' = \langle \phi(Q)\rangle$$

$$\begin{tikzcd}
E \arrow[r, squiggly, "\phi"] \arrow[d, squiggly, "\psi"'] & E/\langle P\rangle \arrow[d, squiggly, "\psi'"] \\
E/\langle Q\rangle \arrow[r, squiggly, "\phi'"'] & E/\langle P, Q\rangle
\end{tikzcd}$$

# Supersingular Isogeny Diffie-Hellman[8]

**Parameters:**

- Prime $p$ such that $p + 1 = \ell_A^a \ell_B^b$;
- Supersingular curve $E \simeq (\mathbb{Z}/(p+1)\mathbb{Z})^2$;
- $E[\ell_A^a] = \langle P_A, Q_A \rangle$;
- $E[\ell_B^b] = \langle P_B, Q_B \rangle$.

**Secret data:**

- $R_A = m_A P_A + n_A Q_A$,
- $R_B = m_B P_B + n_B Q_B$,

$$
\begin{tikzcd}
 & & E \arrow[dll, squiggly, "\phi"'] \arrow[drr, squiggly, "\psi"] & & \\
E/\langle R_A \rangle \arrow[drr, squiggly, "\psi'"'] & & & & E/\langle R_B \rangle \arrow[dll, squiggly, "\phi'"] \\
\frac{E/\langle R_A \rangle}{\phi(R_B)} \simeq & & E/\langle R_A, R_B \rangle & & \simeq \frac{E/\langle R_B \rangle}{\psi(R_A)}
\end{tikzcd}
$$

[8] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

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$$\begin{tikzcd}
 & E \arrow[dl, squiggly, "\phi"'] \arrow[dr, squiggly, "\psi"] & \\
E/\langle R_A \rangle \arrow[dr, squiggly, "\psi'"'] & & E/\langle R_B \rangle \arrow[dl, squiggly, "\phi'"] \\
 & E/\langle R_A, R_B \rangle &
\end{tikzcd}$$

$\phi(P_B)$ $\psi(P_A)$

$\phi(Q_B)$ $\psi(Q_A)$

$$\frac{E/\langle R_A \rangle}{\phi(R_B)} \simeq E/\langle R_A, R_B \rangle \simeq \frac{E/\langle R_B \rangle}{\psi(R_A)}$$

[8] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

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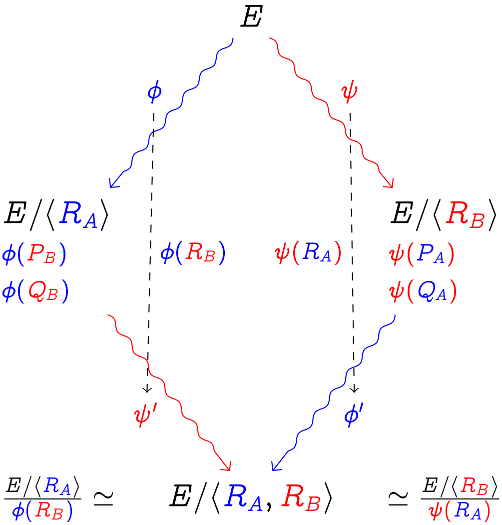



[8] Jao and De Feo 2011; De Feo, Jao, and Plût 2014.

# Performance

- For efficiency choose $p$ such that $p + 1 = 2^a 3^b$.
- For classical $n$-bit security, choose $2^a \sim 3^b \sim 2^{2n}$, hence $p \sim 2^{4n}$.
- For quantum $n$-bit security, choose $2^a \sim 3^b \sim 2^{3n}$, hence $p \sim 2^{6n}$.

## Practical optimizations:

- Use new quasi-linear algorithm for isogeny evaluation[a].
- Optimize arithmetic for $\mathbb{F}_p$.[b,c]
- $-1$ is a quadratic non-residue: $\mathbb{F}_{p^2} \simeq \mathbb{F}_p[X]/(X^2 + 1)$.
- $E$ (or its twist) has a 4-torsion point: use Montgomery form.
- Avoid inversions by using *projective curve equations*.[b]

Fastest implementation[b]: 100Mcycles (Intel Haswell) @128bits quantum security level, 4512bits public key size.

[a] De Feo, Jao, and Plût 2014.
[b] Costello, Longa, and Naehrig 2016.
[c] Karmakar, Roy, Vercauteren, and Verbauwhede 2016.

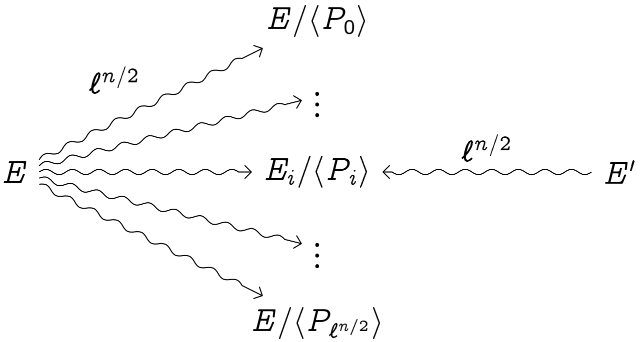
# Comparison

| | Speed | Communication |
|---|---|---|
| RSA 3072 | 4ms | 0.3KiB |
| ECDH nistp256 | 0.7ms | 0.03KiB |
| Code-based | 0.5ms | 360KiB |
| NTRU | 0.3-1.2ms | 1KiB |
| Ring-LWE | 0.2-1.5ms | 2-4KiB |
| LWE | 1.4ms | 11KiB |
| SIDH | 35-400ms | 0.5KiB |

Source: D. Stebila, *Preparing for post-quantum cryptography in TLS*

# Generic attacks

Problem: Given $E, E'$, isogenous of degree $\ell^n$, find $\phi : E \to E'$.




- With high probability $\phi$ is the unique collision (or *claw*).
- A quantum claw finding[9] algorithm solves the problem in $O(\ell^{n/3})$.

[9] Tani 2009.

# Other attacks

## Ephemeral key recovery (total break)

Given $E_0$ and a public curve $E_0/\langle R\rangle$, find the kernel of the secret isogeny:

**Subexponential** $L_p(1/2, \sqrt{3}/2)$ when both curves are defined over $\mathbb{F}_p$.[a]

**Polynomial** isomorphic problem on quaternion algebras.[b]

**Equivalent to** computing the endomorphism rings of both $E_0$ and $E_0/\langle R\rangle$.[c]

[a] Biasse, Jao, and Sankar 2014.
[b] Kohel, K. Lauter, Petit, and Tignol 2014.
[c] Steven D Galbraith, Petit, Shani, and Ti 2016.

**Open problem:** exploit the additional information transmitted by the protocol to improve attacks (classical or quantum).

# Other attacks

## Other security models

**Active attack** against long term keys, learns the full key with (close to) optimal number of oracle queries. Countermeasures are relatively expensive.[a]

**Side channel** Constant-time implementation available.[b]

Attack on partially leaked keys.[c]

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[a] Steven D Galbraith, Petit, Shani, and Ti 2016.
[b] Costello, Longa, and Naehrig 2016.
[c] Gélin and Wesolowski 2017; Ti 2017.

**Open problem:** Create a protocol secure against active adversaries.

# Bonus: a ZK proof of knowledge[10]

Secret: knowledge of the kernel of a degree $\ell_A^{e_A}$ isogeny from $E$ to $E/\langle S\rangle$.

$$E \xrightarrow{\quad\quad\phi\quad\quad} E/\langle S\rangle$$

[10] De Feo, Jao, and Plût 2014.

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Secret: knowledge of the kernel of a degree $\ell_A^{e_A}$ isogeny from $E$ to $E/\langle S\rangle$.

$$\begin{tikzcd}
E \arrow[r, dash, "\phi"] \arrow[d, dash, "?"'] & E/\langle S\rangle \arrow[d, dash, "?"] \\
E/\langle P\rangle \arrow[r, dash, "?"'] & E/\langle P, S\rangle
\end{tikzcd}$$

1. Choose a random point $P \in E[\ell_B^{e_B}]$, compute the diagram;
2. Publish the curves $E/\langle P\rangle$ and $E/\langle P, S\rangle$;

[10] De Feo, Jao, and Plût 2014.

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\end{tikzcd}$$

1. Choose a random point $P \in E[\ell_B^{e_B}]$, compute the diagram;
2. Publish the curves $E/\langle P\rangle$ and $E/\langle P, S\rangle$;
3. The verifier asks one of the two questions:
    - Reveal the degree $\ell_B^{e_B}$ isogenies;

[10] De Feo, Jao, and Plût 2014.

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2. Publish the curves $E/\langle P\rangle$ and $E/\langle P, S\rangle$;
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   - Reveal the degree $\ell_B^{e_B}$ isogenies;
   - Reveal the bottom isogeny.

[10] De Feo, Jao, and Plût 2014.

# Other protocols based on SIDH

## Non-interactive protocols

- El-Gamal encryption.

## Interactive protocols

- Signatures (using Fiat-Shamir)[a],
- Undeniable signatures[b],
- Strong designated verifier signatures[c],
- Authenticated encryption[d].

[a] Steven D Galbraith, Petit, Shani, and Ti 2016.
[b] Jao and Soukharev 2014.
[c] Sun, Tian, and Wang 2012.
[d] Soukharev, Jao, and Seshadri 2016.

**Open problem:** Efficient signatures, …

# Thank you

`http://defeo.lu/`

@luca_defeo

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