How to prove a secret isogeny

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based on joint work with
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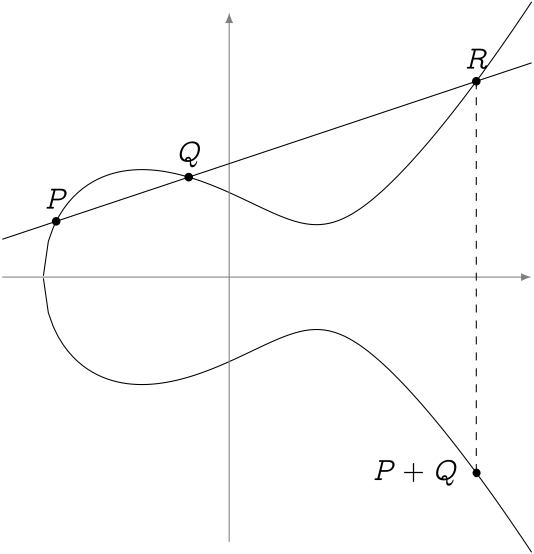
Slides online at https://defeo.lu/docet/

Elliptic curves

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve…



What's scalar multiplication?

$$[n] \; : \; P \mapsto \underbrace{P + P + \cdots + P}_{n \text{ times}}$$

- A map $E \to E$,
- a group morphism,
- with finite kernel
  (the torsion group $E[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$),
- surjective (in the algebraic closure),
- given by rational maps of degree n^2.

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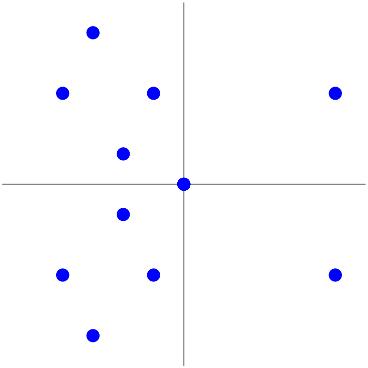
(Separable) isogenies ⇔ finite subgroups:

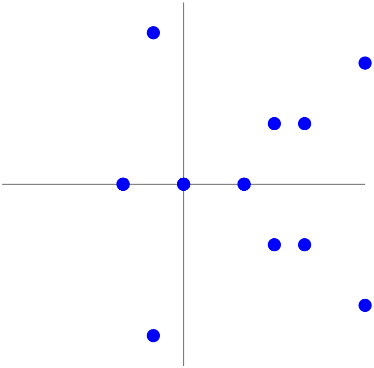
$$0 \to H \to E \xrightarrow{\phi} E' \to 0$$

Isogenies: an example over $\mathbb{F}_{11}$

$$E \;:\; y^2 = x^3 + x$$



$$E' \;:\; y^2 = x^3 - 4x$$



$$\phi(x, y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2} \right)$$

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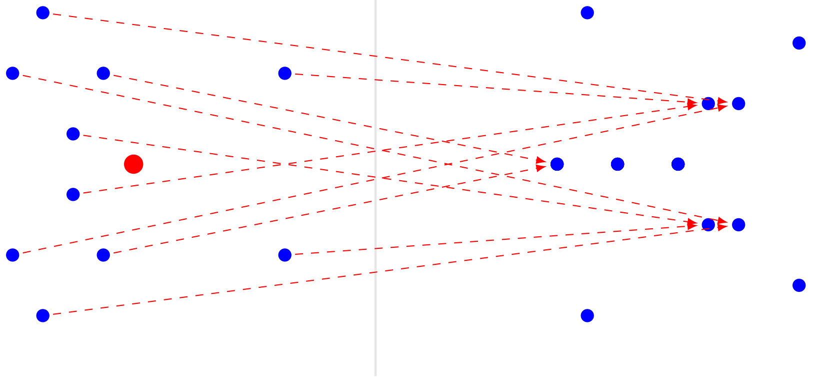
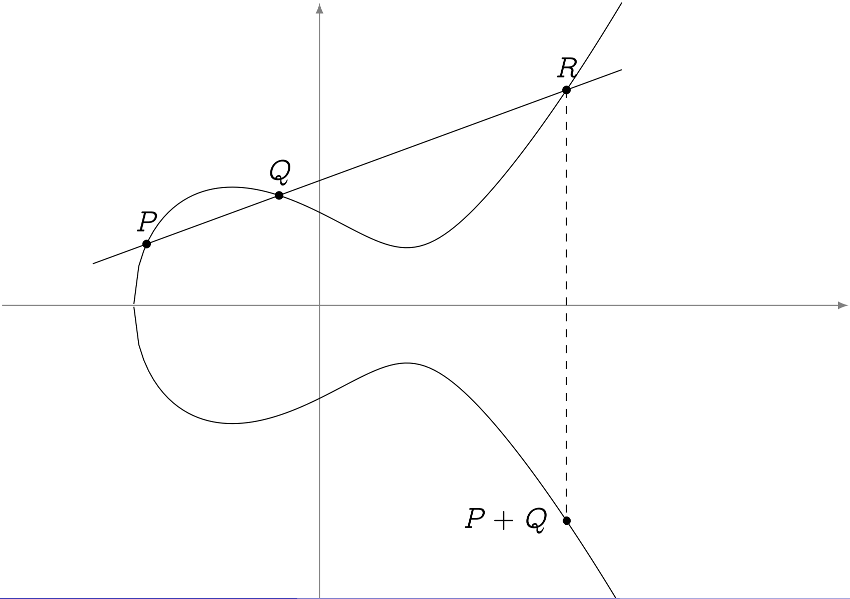


$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \quad y\frac{x^2 - 1}{x^2} \right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Up to isomorphism



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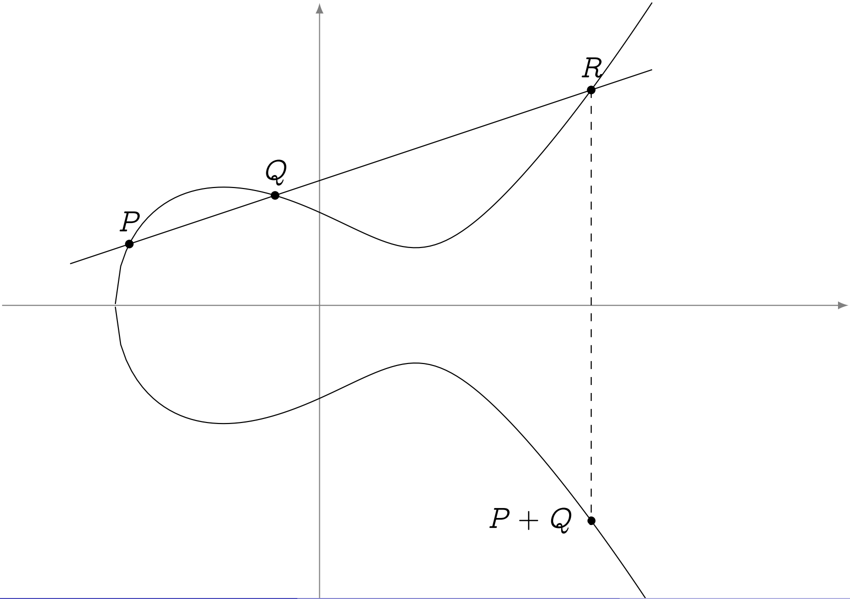

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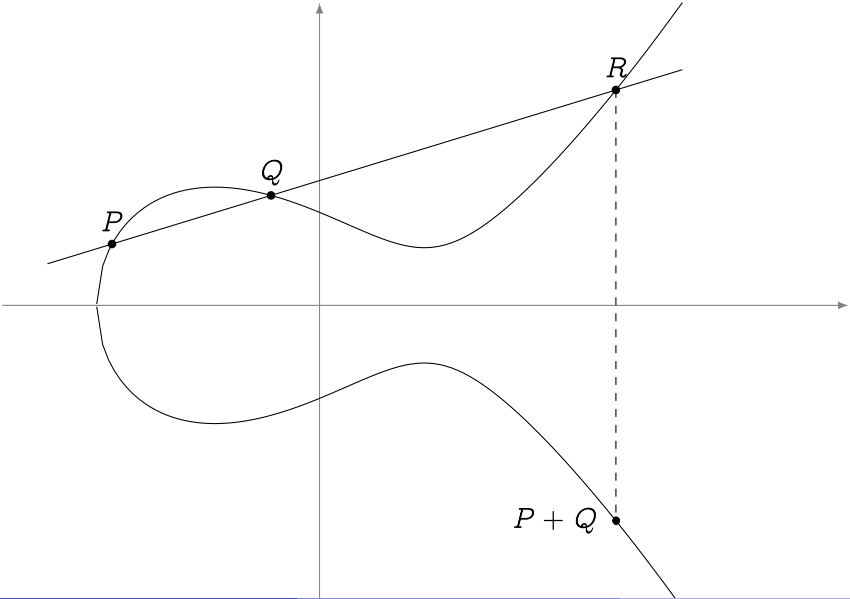

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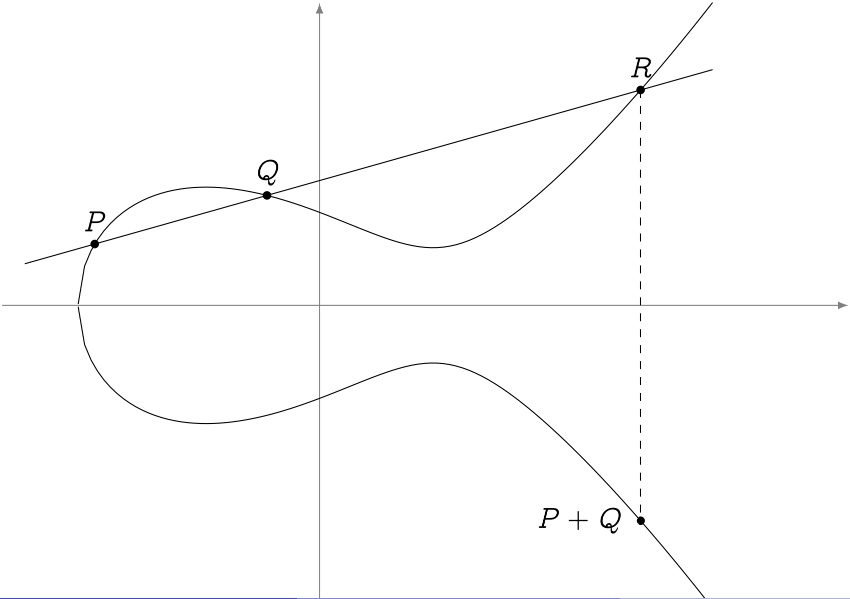

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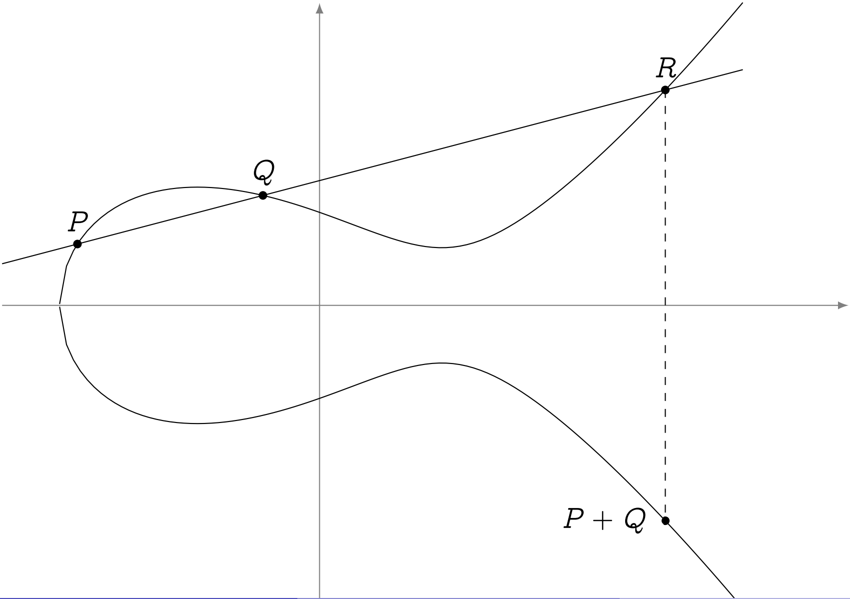

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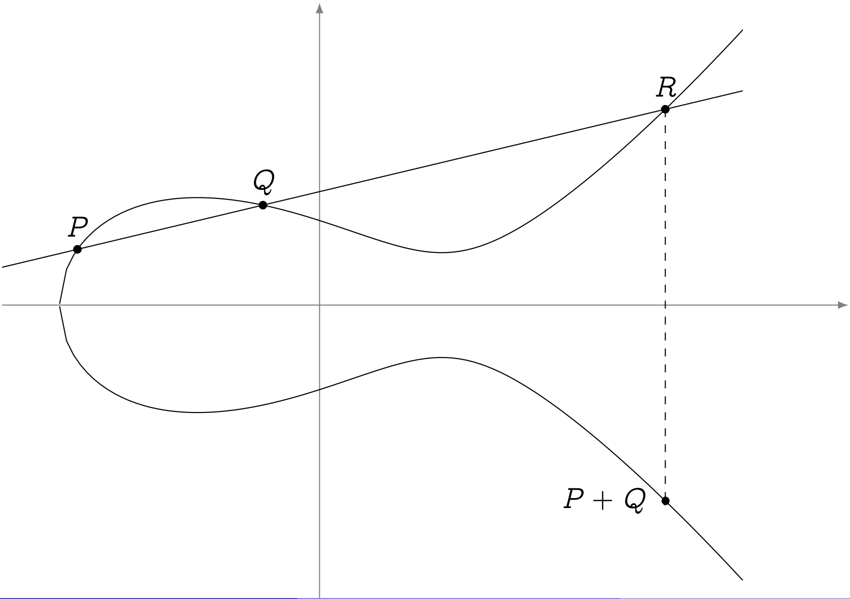

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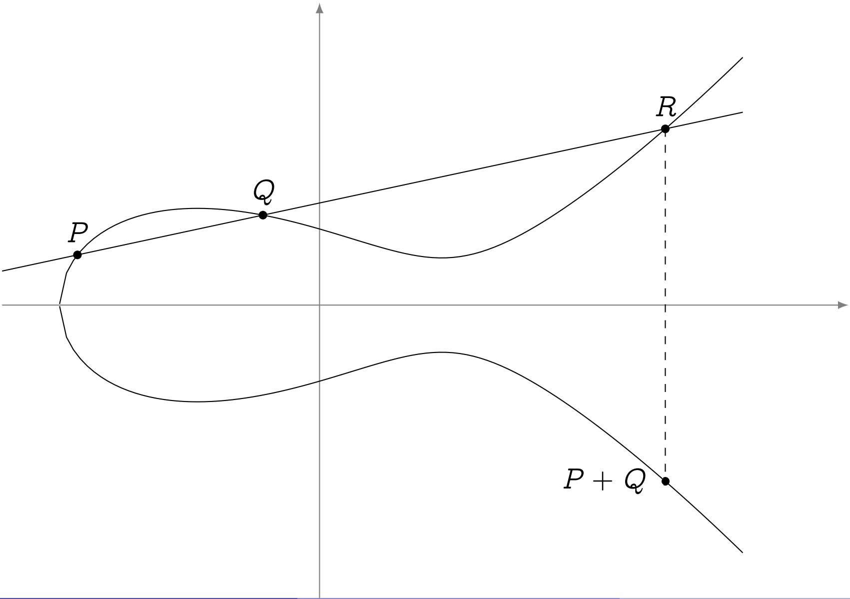

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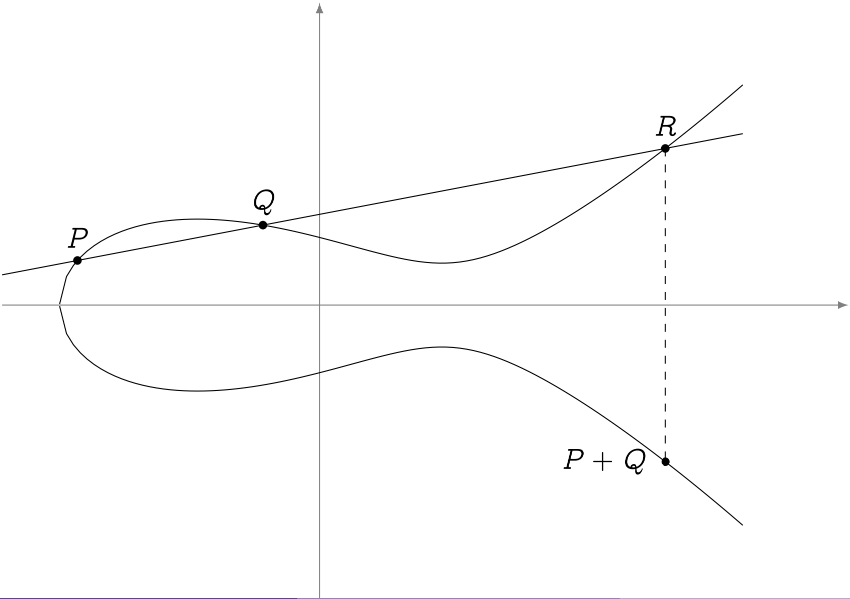

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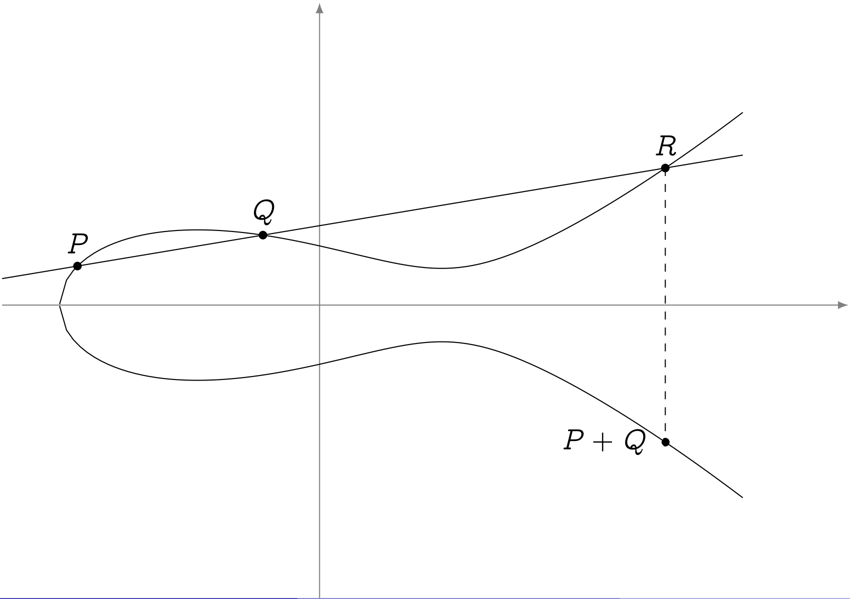

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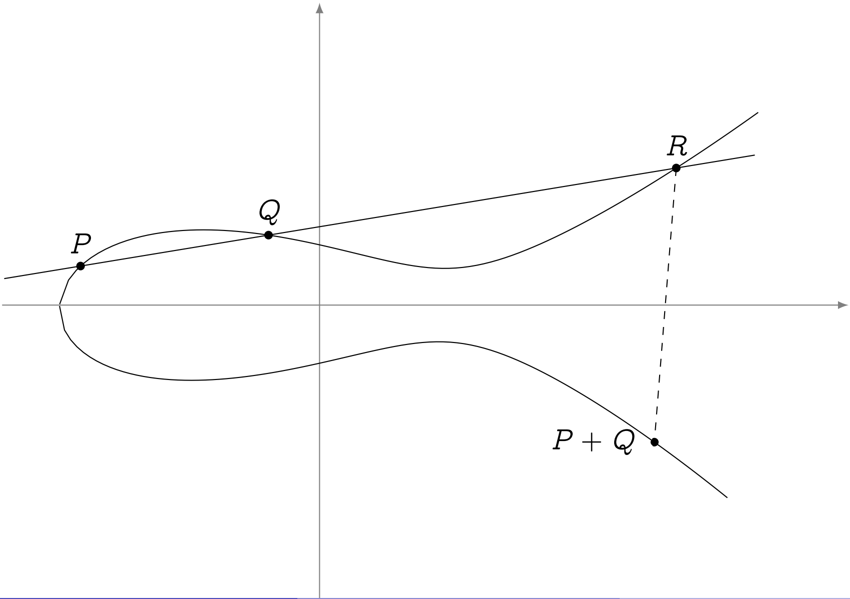

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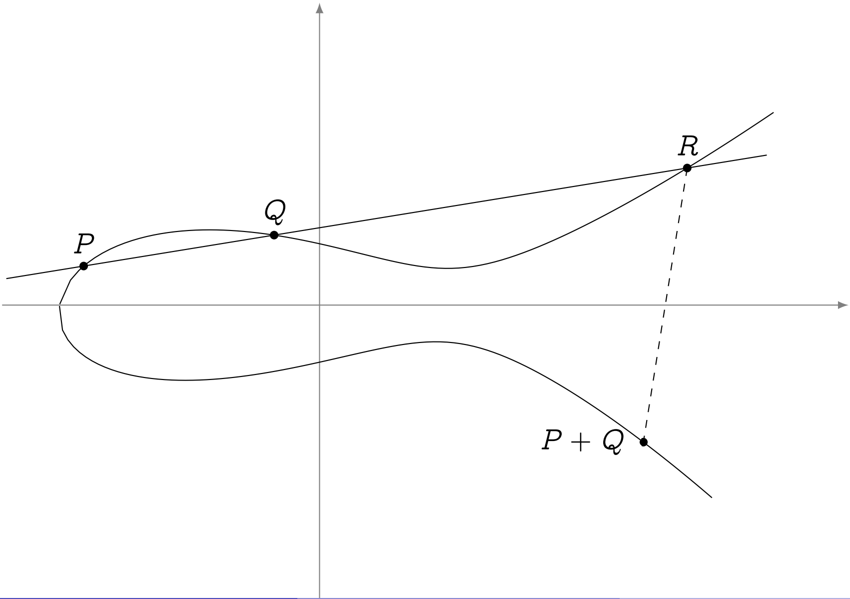

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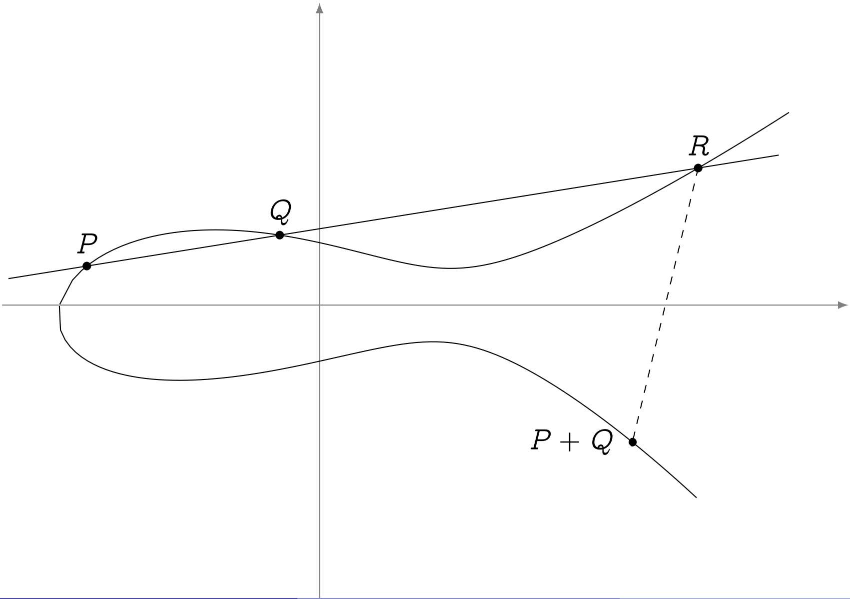

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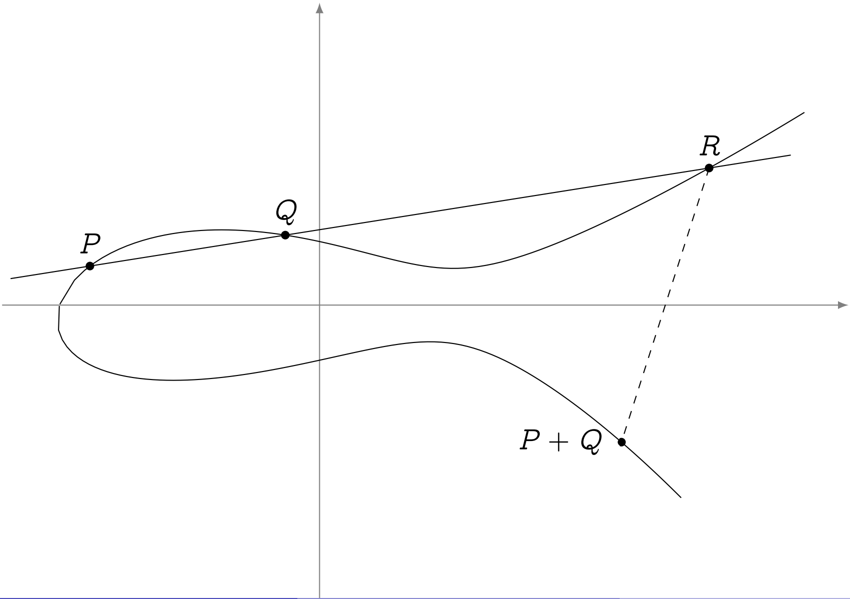

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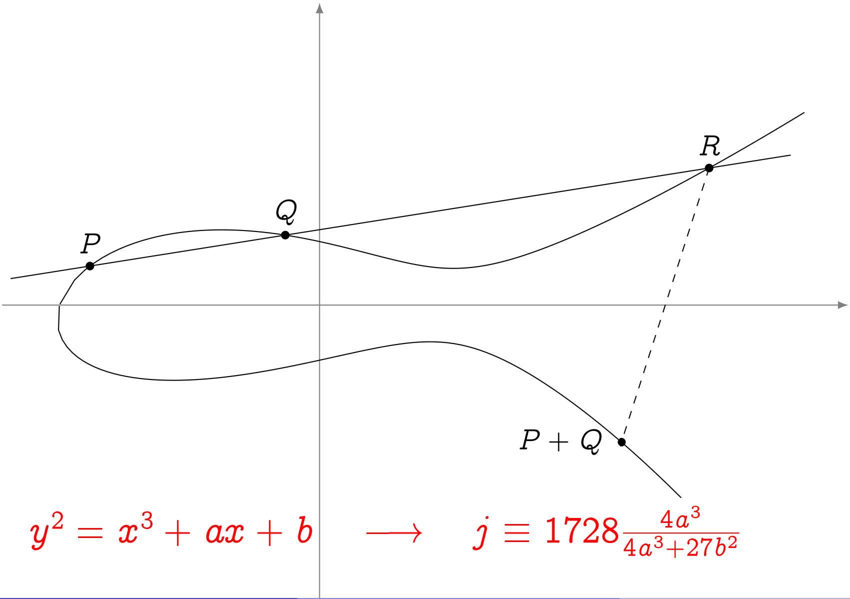
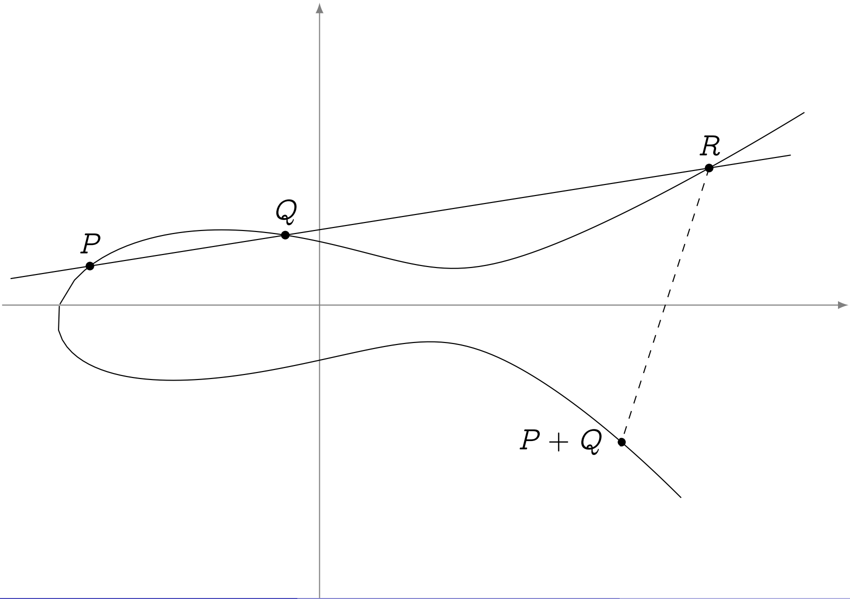


$$y^2 = x^3 + ax + b \quad \longrightarrow \quad j \equiv 1728\frac{4a^3}{4a^3+27b^2}$$

Up to isomorphism



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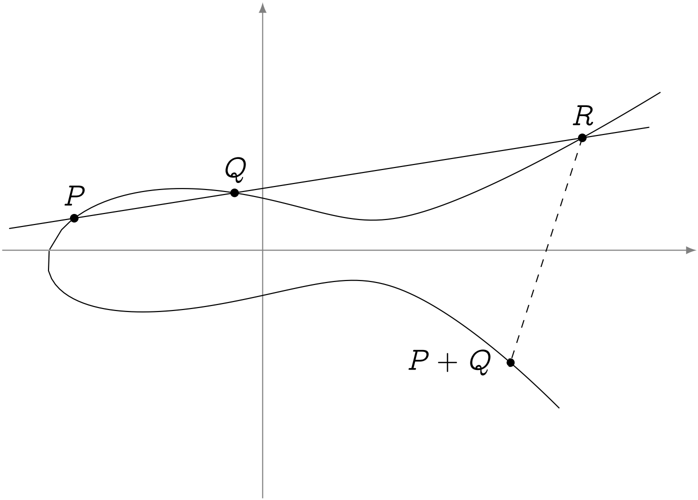

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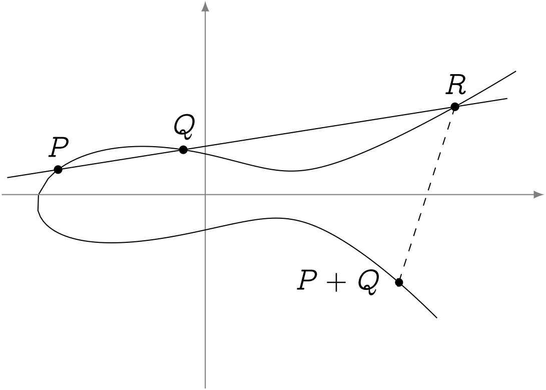

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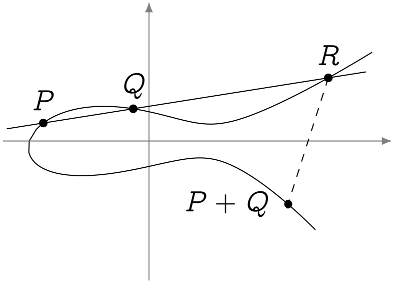

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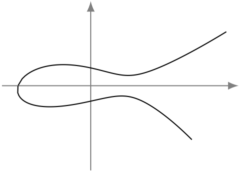

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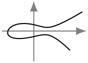


Up to isomorphism

$j = 1728$

Up to isomorphism

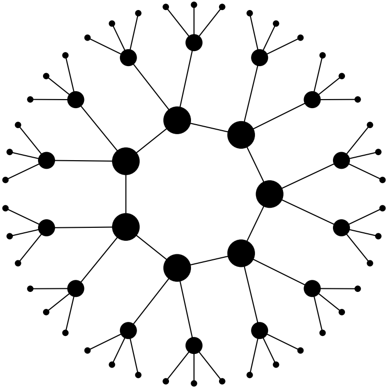
$$j = 1728 \xrightarrow{\phi} j = 287496$$

Isogeny graphs

We look at the graph of elliptic curves with isogenies up to isomorphism. We say two isogenies ϕ, ϕ' are isomorphic if:

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & E' \arrow[d, leftrightarrow, "\wr"] \\ & E' \end{tikzcd}$$

Example: Finite field, ordinary case, graph of isogenies of degree 3.



The graph of isogenies of prime degree $\ell \neq p$

All graphs are undirected (dual isogeny theorem).

Ordinary case (isogeny volcanoes)

- Nodes can have degree $0, 1, 2$ or $\ell + 1$.
  - For $\sim 50\%$ of the primes ℓ, graphs are just isolated points;
  - For other $\sim 50\%$, graphs are 2-regular;
  - other cases only happen for finitely many ℓ's.

Supersingular case ($\mathbb{F}_p$)

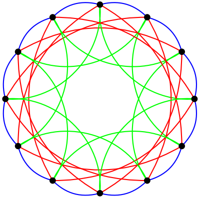
- If $\ell = 2$ nodes have degree $1, 2$ or 3;
- For $\sim 50\%$ of ℓ, graphs are isolated points;
- For other $\sim 50\%$, graphs are 2-regular;

Supersingular case ($\mathbb{F}_{p^2}$)

- The graph is $\ell + 1$-regular.
- There is a unique (finite) connected component made of all supersingular curves with the same number of points.

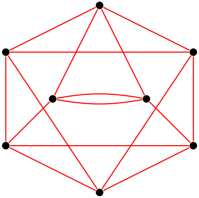
Isogeny graphs taxonomy

Complex Multiplication (CM) graphs



- Ordinary / Supersingular ($\mathbb{F}_p$)
- Superposition of isogeny cycles (one color per degree)
- Isomorphic to Cayley graph of a quadratic class group
- Large automorphism group
- Typical size $O(\sqrt{p})$
- Used in: **CSIDH**

Full **supersingular graphs**



- Supersingular ($\mathbb{F}_{p^2}$)
- One isogeny degree
- $(\ell + 1)$-regular
- Tiny automorphism group
- Size $\approx p/12$
- Used in: **SIDH**

Post-quantum isogeny primitives

SIDH (Jao, De Feo 2011)

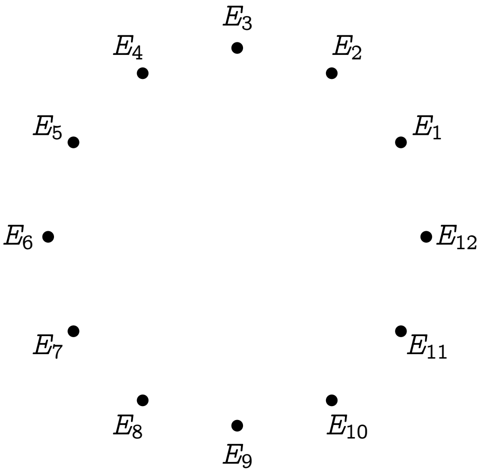
- Pronounce S–I–D–H;
- Based on isogeny walks in the full supersingular graph over $\mathbb{F}_{p^2}$;
- Basis for the NIST KEM candidate SIKE;
- Better asymptotic quantum security;
- Short keys, slow.

CSIDH (Couveignes 1996; Rostovtsev, Stolbunov 2006; Castryck, Lange, Martindale, Panny, Renes 2018)

- Pronounce Sea–Side;
- Based on isogeny walks in the supersingular CM graph over $\mathbb{F}_p$;
- Straightforward generalization of Diffie–Hellman;
- More “natural” security assumption;
- Shorter keys, slower.

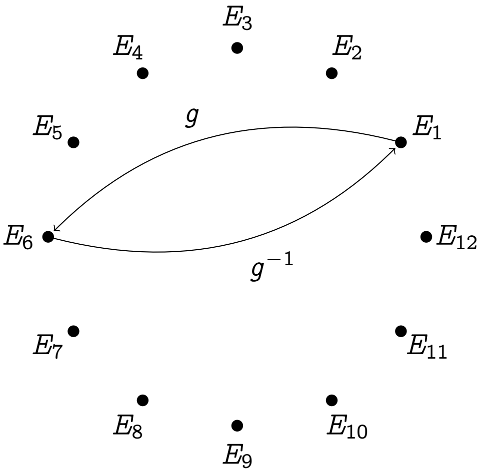
CSIDH key exchange

- A set of supersingular elliptic curves over $\mathbb{F}_p$;



CSIDH key exchange

- A set of supersingular elliptic curves over $\mathbb{F}_p$;
- A group action by a commutative class group G;



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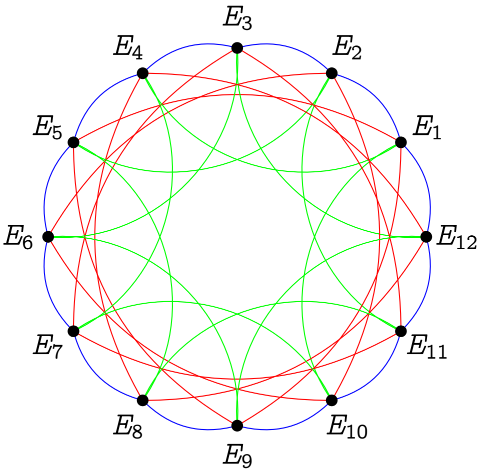

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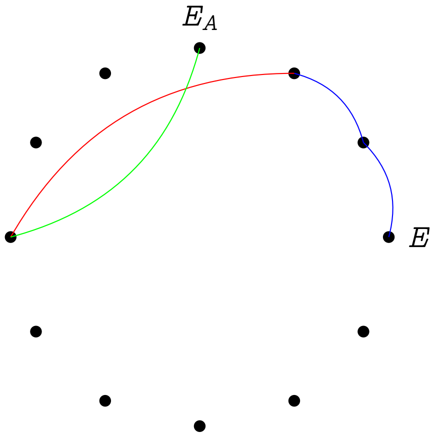
Key exchange:

- Alice picks secret $a = g_2^{a_2} g_3^{a_3} g_5^{a_5} \cdots$,



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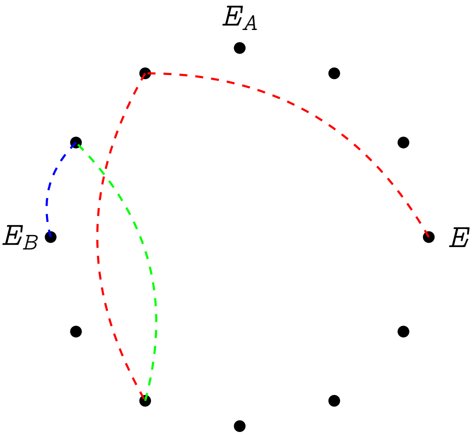

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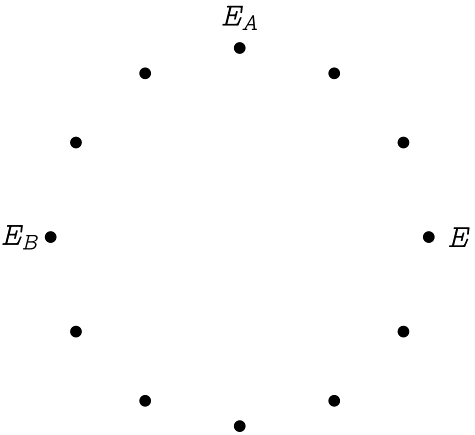
- They exchange $E_A = a * E_1$ and $E_B = b * E_1$,



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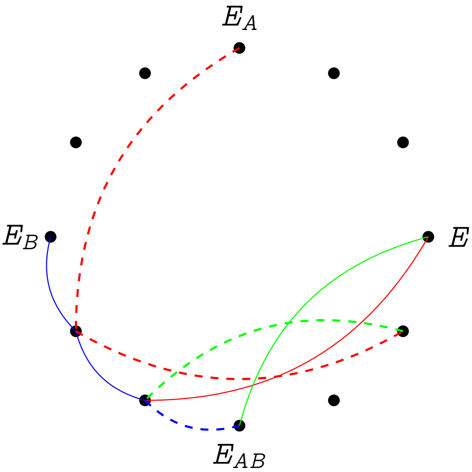
- They exchange $E_A = a * E_1$ and $E_B = b * E_1$,
- Shared secret is $E_{AB} = (ab) * E_1 = a * E_B = b * E_A$.



SIDH key exchange

Good news: there is no action of a commutative class group.

Bad news: there is no action of a commutative class group.

Idea: Let Alice and Bob walk in two different isogeny graphs on the same vertex set.

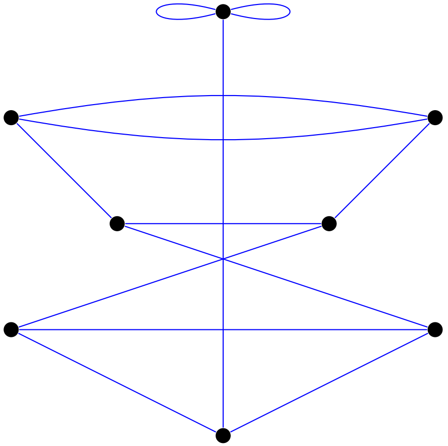


Figure: 2- and 3-isogeny graphs on $\mathbb{F}_{97^2}$.

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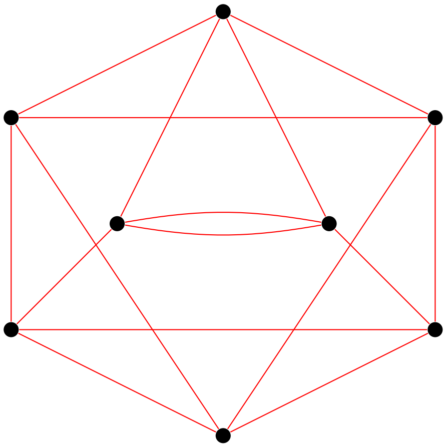

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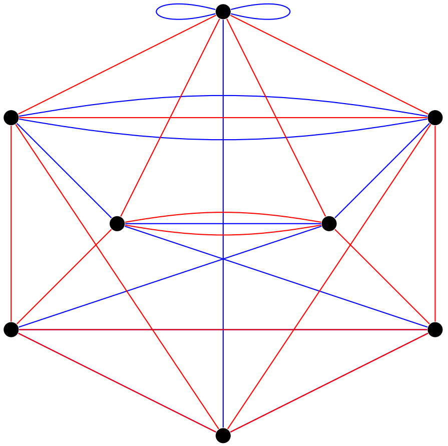

Figure: 2- and 3-isogeny graphs on $\mathbb{F}_{97^2}$.

SIDH key exchange

- Fix small primes ℓ_A, ℓ_B;
- No canonical labeling of the ℓ_A- and ℓ_B-isogeny graphs; however…

Walk of length e_A

$=$

Isogeny of degree $\ell_A^{e_A}$

$=$

Kernel $\langle P\rangle \subset E[\ell_A^{e_A}]$

$$\ker\phi = \langle P\rangle \subset E[\ell_A^{e_A}]$$

$$\ker\psi = \langle Q\rangle \subset E[\ell_B^{e_B}]$$

$$\ker\phi' = \langle\psi(P)\rangle$$

$$\ker\psi' = \langle\phi(Q)\rangle$$

$$\begin{tikzcd}
E \arrow[r, squiggly, "\phi"] \arrow[d, squiggly, "\psi"'] & E/\langle P\rangle \arrow[d, squiggly, "\psi'"] \\
E/\langle Q\rangle \arrow[r, squiggly, "\phi'"'] & E/\langle P, Q\rangle
\end{tikzcd}$$

Security assumptions

Isogeny walk problem

Input Two isogenous elliptic curves E, E' over $\mathbb{F}_q$.

Output A path $E \to E'$ in an isogeny graph.

SIDH problem (1)

Input Elliptic curves E, E' over $\mathbb{F}_q$, isogenous of degree $\ell_A^{e_A}$.

Output The unique path $E \to E'$ of length e_A in the ℓ_A-isogeny graph.

SIDH problem (2)

Input
- Elliptic curves E, E' over $\mathbb{F}_q$, isogenous of degree $\ell_A^{e_A}$;
- The action of the isogeny on $E[\ell_B^{e_B}]$.

Output The unique path $E \to E'$ of length e_A in the ℓ_A-isogeny graph.

Why prove a secret isogeny?

Public: Curves E, E'

Secret: An isogeny walk $E \to E'$

Why?

- For interactive identification;
- For signing messages;
- For validating public keys (esp. SIDH);
- More…

Some properties

	Zero knowledge			
	Statistical	Computational	Quantum resistance	Succinctness
CSIDH	✓		✓	
SIDH		✓	✓	
Pairings				✓

A Σ-protocol from Diffie–Hellman[1]

- A key pair (s, g^s);

$$g \xrightarrow{\quad s \quad} g^s$$

[1] Kids, do not try this at home! Use Schnorr!

A Σ-protocol from Diffie–Hellman[1]

- A key pair (s, g^s);
- Commit to a random element g^r;

$$
\begin{tikzcd}
g \arrow[rr, "s"] \arrow[dr, "r"'] & & g^s \\
& g^r &
\end{tikzcd}
$$

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- A key pair (s, g^s);
- Commit to a random element g^r;
- Challenge with bit $b \in \{0, 1\}$;

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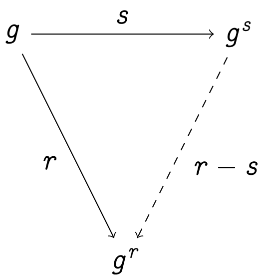

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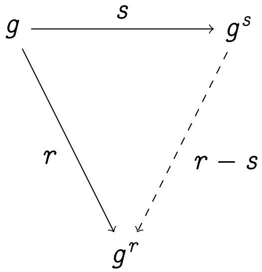
- Challenge with bit $b \in \{0, 1\}$;
- Respond with $c = r - b \cdot s \mod \#G$;
- Verify that $g^c(g^s)^b = g^r$.

$$\begin{tikzcd}
g \arrow[rr, "s"] \arrow[dr, "r"'] & & g^s \arrow[dl, dashed, "r - s"] \\
& g^r &
\end{tikzcd}$$

[1] Kids, do not try this at home! Use Schnorr!

A Σ-protocol from Diffie–Hellman[1]

- A key pair (s, g^s);
- Commit to a random element g^r;
- Challenge with bit $b \in \{0, 1\}$;
- Respond with $c = r - b \cdot s \mod \#G$;
- Verify that $g^c (g^s)^b = g^r$.



Zero-knowledge

Does not leak because:
c is uniformly distributed and independent from s.

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Zero-knowledge

Does not leak because:
c is uniformly distributed and
independent from s.

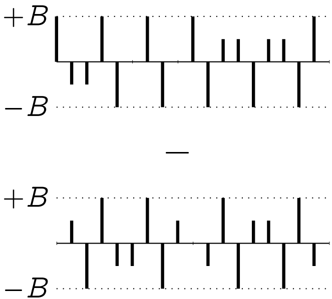
Unlike Schnorr, compatible with
group action Diffie–Hellman.

[1] Kids, do not try this at home! Use Schnorr!

The trouble with groups of unknown structure

In CSIDH secrets look like:
$g^{\vec{s}} = g_2^{s_2} g_3^{s_3} g_5^{s_5} \cdots$

- the elements g_i are fixed,
- the secret is the exponent vector $\vec{s} = (s_2, s_3, \dots) \in [-B, B]^n$,
- secrets must be sampled in a box $[-B, B]^n$ "large enough"…



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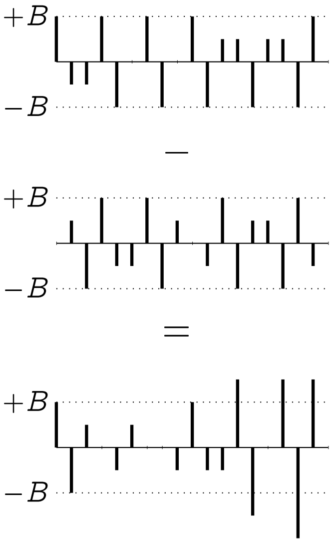


The leakage

With $\vec{s}, \vec{r} \xleftarrow{\$} [-B, B]^n$, the distribution of $\vec{r} - \vec{s}$ depends on the long term secret $\vec{s}$!

The two fixes

Compute the group structure and stop whining

CSI-FiSh: Beullens, Kleinjung and Vercauteren 2019 (eprint:2019/498)

- Already suggested by Couveignes (1996) and Stolbunov (2006).
- Computationally intensive (subexponential parameter generation).
- Decent parameters, e.g.: 263 bytes, 390 ms, @NIST-1.
- – Technically not post-quantum.

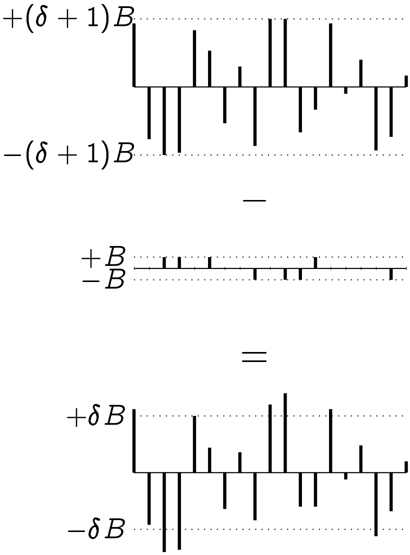
Do like the lattice people

SeaSign: D. and Galbraith 2019

- Use Fiat–Shamir with aborts (Lyubashevsky 2009).
- – Huge increase in signature size and time.
- Compromise signature size/time with public key size (still slow).

Rejection sampling

- Sample long term secret $\vec{s}$ in the usual box $[-B, B]^n$,
- Sample ephemeral $\vec{r}$ in a larger box $[-(\delta+1)B, (\delta+1)B]^n$,
- Throw away $\vec{r} - \vec{s}$ if it is out of the box $[-\delta B, \delta B]^n$.



Zero-knowledge

Theorem: $\vec{r} - \vec{s}$ is uniformly distributed in $[-\delta B, \delta B]^n$.

Problem: set δ so that rejection probability is low.

Performance

- For λ-bit security, protocol must be repeated λ times in parallel;
- $\delta = \lambda n$ for a rejection probability $\leq 1/3$;
- Signature size $\approx \lambda n$ coefficients $\in [-\delta B, \delta B]$;
- Sign/verify time linear in $\|\vec{r} - \vec{s}\|_\infty \approx \lambda^2 n^2 B$.

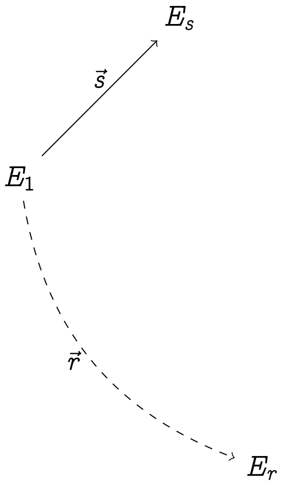
CSIDH instantiation (NIST-1)

Parameters:	$\lambda = 128$, $n = 74$, $B = 5$;
PK size:	64 B
SK size:	32 B
Signature:	20 KiB
Verify time:	10 hours
Sign time:	$3\times$ verify

Key/signature size compromise

- One key pair $(\vec{s}, E_s)$;
- Challenge $b \in \{0, 1\}$;
- Reveal $\vec{r} - b\vec{s}$;

$\rightarrow$ λ iterations;



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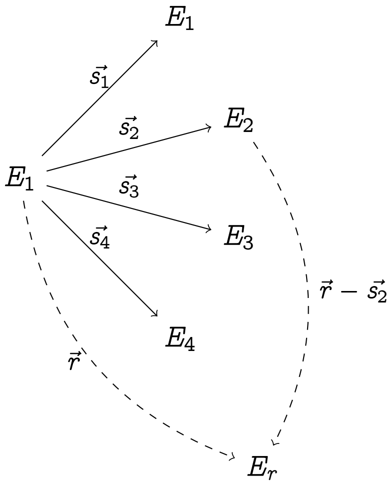
Compromise: $\mathbf{t}$-bit challenges

- $\mathbf{2^t}$ key pairs $(\vec{s_i}, E_i)$;
- Challenge $b \in \{0, 2^t\}$;
- Reveal $\vec{r} - \vec{s_b}$;

$\rightarrow$ $\lambda/\mathbf{t}$ iterations;



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$\rightarrow$ Sample $r \xleftarrow{\$} [-\lambda nB, \lambda nB]$.

Compromise: t-bit challenges

- 2^t key pairs $(\vec{s_i}, E_i)$;
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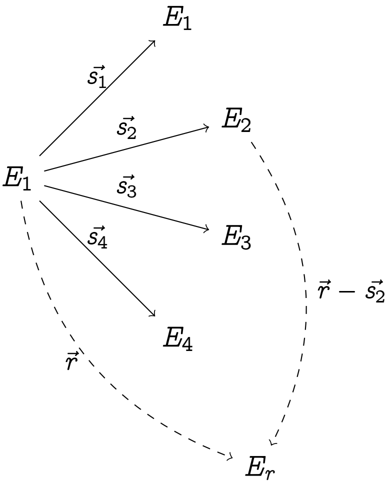
- Reveal $\vec{r} - \vec{s_b}$;

$\rightarrow$ λ/t iterations;



Key/signature size compromise

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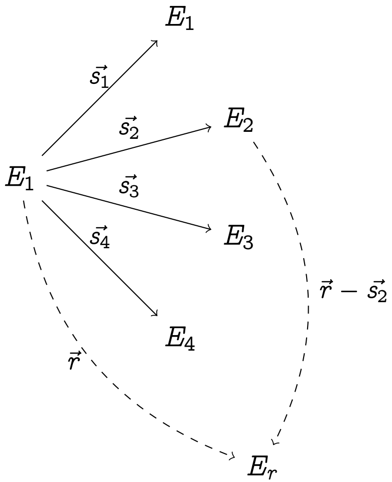
$\rightarrow$ Sample $r \xleftarrow{\$} [-\lambda nB, \lambda nB]$.

Compromise: $\mathbf{t}$-bit challenges

- $\mathbf{2^t}$ key pairs $(\vec{s_i}, E_i)$;
- Challenge $b \in \{0, 2^t\}$;
- Reveal $\vec{r} - \vec{s_b}$;

$\rightarrow$ $\lambda/\mathbf{t}$ iterations;

$\rightarrow$ Sample $r \xleftarrow{\$} [-\lambda nB/\mathbf{t}, \lambda nB/\mathbf{t}]$.



Public key compression

$$
\begin{tikzcd}[row sep=small]
 & E_1 \\
 & E_2 \\
E_1 \arrow[ruu] \arrow[ru] \arrow[rd] \arrow[rdd] & \\
 & E_3 \\
 & E_4
\end{tikzcd}
$$

Public key compression

$$
\begin{tikzcd}
& E_1 \arrow[r] & H(E_1) \arrow[dr] & & \\
& & & H(\bullet,\bullet) \arrow[ddr] & \\
& E_2 \arrow[r] & H(E_2) \arrow[ur] & & \\
E_1 \arrow[uuur] \arrow[ur] \arrow[dr] \arrow[dddr] & & & & H(\bullet,\bullet) = pk \\
& E_3 \arrow[r] & H(E_3) \arrow[dr] & & \\
& & & H(\bullet,\bullet) \arrow[uur] & \\
& E_4 \arrow[r] & H(E_4) \arrow[ur] & &
\end{tikzcd}
$$

- Construct Merkle tree on top of public keys, root is the new public key;

Public key compression

$$\begin{tikzcd}
 & E_1 \arrow[r] & H(E_1) \arrow[dr] & & \\
 & & & H(\bullet,\bullet) \arrow[ddr] & \\
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E_1 \arrow[uuur] \arrow[ur] \arrow[dr] \arrow[dddr] & & & & H(\bullet,\bullet) = pk \\
 & E_3 \arrow[r] & H(E_3) \arrow[dr] & & \\
 & & & H(\bullet,\bullet) \arrow[uur] & \\
 & E_4 \arrow[r] & H(E_4) \arrow[ur] & &
\end{tikzcd}$$

- Construct Merkle tree on top of public keys, root is the new public key;
- Include Merkle proof in the signature.

SeaSign Performance (NIST-1)

	$t = 1$ **bit challenges**	$t = 16$ **bits challenges**	**PK compression**
Sig size	20 KiB	978 B	3136 B
PK size	64 B	4 MiB	32 B
SK size	32 B	16 B	1 MiB
Est. keygen time	30 ms	30 mins	30 mins
Est. sign time	30 hours	6 mins	6 mins
Est. verify time	10 hours	2 mins	2 mins
Asymptotic sig size	$O(\lambda^2 \log(\lambda))$	$O(\lambda t \log(\lambda))$	$O(\lambda^2 t)$

Recent speed/size compromises by Decru, Panny and Vercauteren

Sig size	36 KiB	2 KiB	—
Est. sign time	30 mins	80 s	—
Est. verify time	20 mins	20 s	—

A Σ-protocol for SIDH

$$E \xrightarrow{\phi} E/\langle S\rangle$$

$\frac{1}{3}$-soundness

Secret ϕ of degree $\ell_A^{e_A}$.

A Σ-protocol for SIDH

$$
\begin{tikzcd}
E \arrow[r, dash, "\phi"] \arrow[d, dash, "?"'] & E/\langle S\rangle \arrow[d, dash, "?"] \\
E/\langle P\rangle \arrow[r, dash, "?"'] & E/\langle P, S\rangle
\end{tikzcd}
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$\frac{1}{3}$-soundness

Secret ϕ of degree $\ell_A^{e_A}$.

1. Choose a random point $P \in E[\ell_B^{e_B}]$, compute the diagram;
2. Publish the curves $E/\langle P\rangle$ and $E/\langle P, S\rangle$;

A Σ-protocol for SIDH

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\begin{tikzcd}
E \arrow[r, dash, "\phi"] \arrow[d, dash, "\psi"'] & E/\langle S\rangle \arrow[d, dash, "?"] \\
E/\langle P\rangle \arrow[r, dash, "?"'] & E/\langle P, S\rangle
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   - Isogenies ψ, ψ' (degree $\ell_B^{e_B}$) unrelated to secret;

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A Σ-protocol for SIDH

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Improving to $\frac{1}{2}$-soundness

- Reveal ψ, ψ' simultaneously;
- Reveals action of ϕ on $E[\ell_B^{e_B}]$ $\Rightarrow$ Stronger security assumption.

SIDH signature performance (NIST-1)

According to Yoo, Azarderakhsh, Jalali, Jao and Vladimir Soukharev 2017:

Size: $\approx 100KB$,

Time: seconds.

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Size: $\approx 100KB$,

Time: seconds.

Galbraith, Petit and Silva 2017

- Concept similar to CSI-FiSh: exploits known structure of endomorphism ring;
- Statistical zero knowledge (under heuristic assumptions);
- Based on the generic isogeny walk problem (requires special starting curve, though);
- Size/performance comparable to Yoo *et al.* (and possibly slower).

Weil pairing and isogenies

Theorem

Let $\phi : E \to E'$ be an isogeny and $\hat\phi : E' \to E$ its dual.
Let e_N be the Weil pairing of E and e'_N that of E'. Then, for

$$e_N(P, \hat\phi(Q)) = e'_N(\phi(P), Q),$$

for any $P \in E[N]$ and $Q \in E'[N]$.

Corollary

$$e'_N(\phi(P), \phi(Q)) = e_N(P, Q)^{\deg \phi}.$$

Refresher: Boneh–Lynn–Shacham (BLS) signatures

Setup:

- Elliptic curve $E/\mathbb{F}_p$, s.t $N|\#E(\mathbb{F}_p)$ for a large prime N,
- (Weil) pairing $e_N : E[N] \times E[N] \to \mathbb{F}_{p^k}$ for some small embedding degree k,
- A decomposition $E[N] = X_1 \times X_2$, with $X_1 = \langle P \rangle$.
- A hash function $H : \{0,1\}^* \to X_2$.

Private key: $s \in \mathbb{Z}/N\mathbb{Z}$.

Public key: sP.

Sign: $m \mapsto sH(m)$.

Verifiy: $e_N(P, sH(m)) = e_N(sP, H(m))$.

$$\begin{tikzcd}
X_1 \times X_2 \arrow[r, "{[s] \times 1}"] \arrow[d, "{1 \times [s]}"'] & X_1 \times X_2 \arrow[d, "e_N"] \\
X_1 \times X_2 \arrow[r, "e_N"'] & \mathbb{F}_{p^k}
\end{tikzcd}$$

US patent 8,250,367 (Broker, Charles and Lauter 2012)

Signatures from isogenies + pairings

- Replace the secret $[s] : E \to E$ with an isogeny $\phi : E \to E'$;
- Define decompositions

$$E[N] = X_1 \times X_2, \qquad E'[N] = Y_1 \times Y_2,$$

  s.t. $\phi(X_1) = Y_1$ and $\phi(X_2) = Y_2$;
- Define a hash function $H : \{0,1\}^* \to Y_2$.

$$\begin{tikzcd}
X_1 \times Y_2 \arrow[r, "\phi \times 1"] \arrow[d, "1 \times \hat{\phi}"'] & Y_1 \times Y_2 \arrow[d, "e'_N"] \\
X_1 \times X_2 \arrow[r, "e_N"'] & \mathbb{F}_{p^k}
\end{tikzcd}$$

Pairing proofs: what for?

- Non-interactive, not post-quantum, not zero knowledge;
- Useful for (partially) validating SIDH public keys;
- Succinct: proof size, verification time independent of walk length!

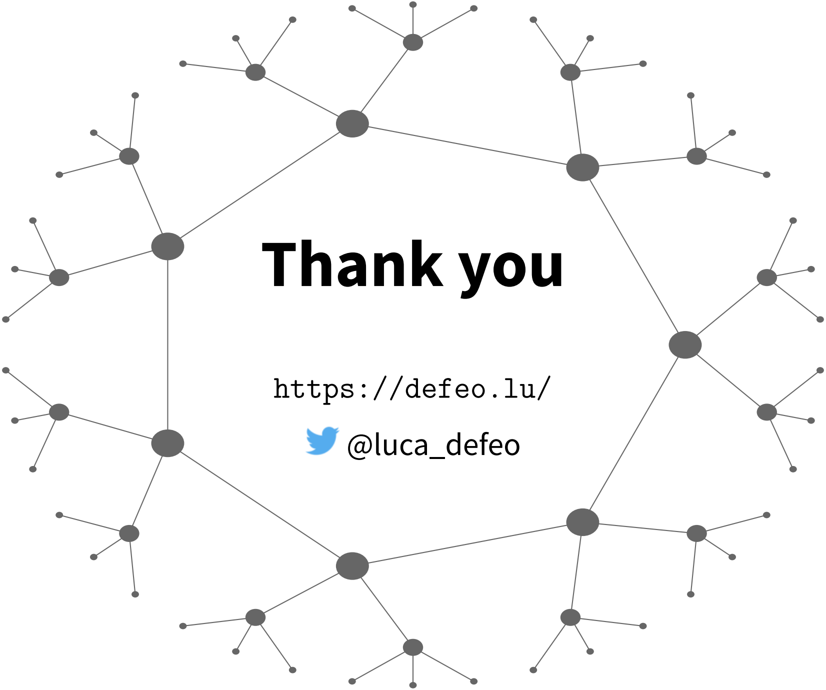
Application: Verifiable Delay Functions

D., Masson, Petit and Sanso 2019 (eprint:2019/166):

- Similar to time-lock puzzles;
- No secret: everything is public;
- Generating proof takes configurable *sequential* time T;
- Verifying proof takes time independent from T;
- Security assumptions very different and new!
- Applications to blockchains: randomness beacons, consensus protocols, …

Conclusion

- Different isogeny graphs enable different styles of proofs, different security assumptions.
- Post-quantum isogeny signatures are still far from practical.
- Practical isogeny signatures do exists (CSI-FiSh); you can start using them now if you are an isogeny hippie, but they do not scale.
- Pairing-based proofs are usable, but not interesting for signatures: look into succinctness, instead!
- Tons of open questions on classical and quantum security, on security proofs, and on constructions.
- Proofs can be chained easily: useful for multi-party supersingular curve generation (work in progress with J. Burdges).
- The isogenista dream: a one-pass post-quantum signature scheme based on walks in isogeny graphs.



Thank you

https://defeo.lu/

@luca_defeo