$\sqrt{\text{élu's formulas}}$

Faster Evaluation of Isogenies of Large Prime Degree

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joint work with D.J. Bernstein, A. Leroux, B. Smith

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Why isogenies?

Six families still in NIST post-quantum competition:

Lattices	9 encryption	3 signature
Codes	7 encryption	
Multivariate		4 signature
Isogenies	1 encryption	
Hash-based		1 signature
MPC		1 signature

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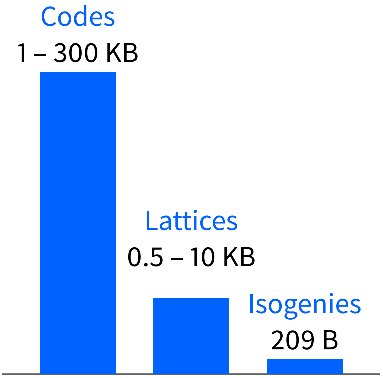

Public key size

NIST-1 level (AES128)

(not to scale)

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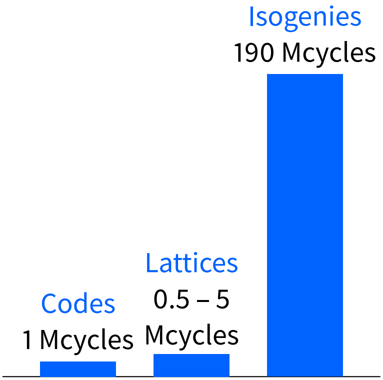

Encryption performance

NIST-1 level (AES128)

(not to scale)

Iso-what?!

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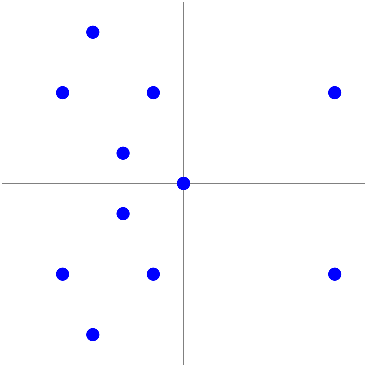
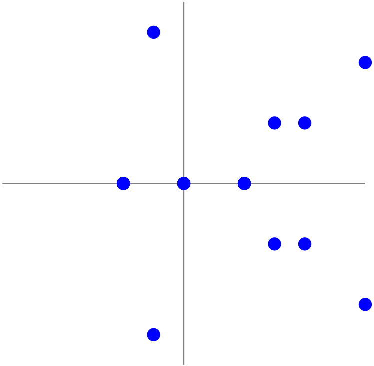
- It is entirely determined by its kernel (i.e., by a single point);
- Isogeny degree $=$ size of the kernel $=$ order of kernel generator $\approx$ size of the polynomials;

Isogenies: an example over $\mathbb{F}_{11}$

$E : y^2 = x^3 + x$



$E' : y^2 = x^3 - 4x$



$$\phi(x, y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2}\right)$$

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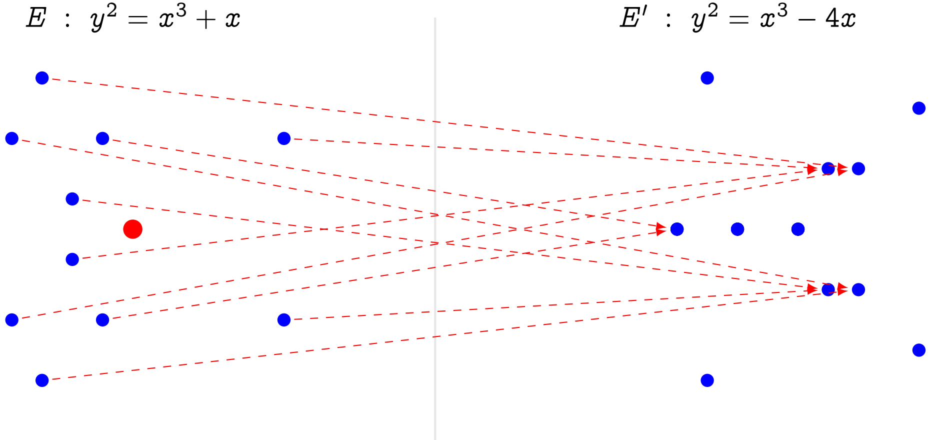

$$\phi(x, y) = \left(\frac{x^2+1}{x}, \quad y\frac{x^2-1}{x^2}\right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Algebraic complexity

In this talk, k is an arbitrary field.

All complexities are in terms of operations over k.

Anatomy of an isogeny (short Weierstrass form)

$$\phi(x, y) = \left(\frac{x^4 - x^3 + 11x^2 + 9x + 12}{x^3 - x^2 - x + 1},\ y\frac{x^5 - x^4 - 14x^3 - 26x^2 - 67x - 21}{x^5 - x^4 - 2x^3 + 2x^2 + x - 1} \right)$$

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degree (the exponent 4 of the numerator)

degree -1 (the exponent 3 of the denominator)

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kernel polynomial

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Point of order 2

Two points of order 4

Anatomy of an isogeny (short Weierstrass form)

degree

$$\phi(x,y) = \left(\frac{x^4 - x^3 + 11x^2 + 9x + 12}{h(x)},\; y\frac{x^5 - x^4 - 14x^3 - 26x^2 - 67x - 21}{x^5 - x^4 - 2x^3 + 2x^2 + x - 1}\right)$$

$$h(x) = \prod_{P\in K\setminus\{0\}} \bigl(x - x(P)\bigr)$$

Anatomy of an isogeny (short Weierstrass form)

computed by Vélu–Elkies–Kohel formulas

$$\phi(x,y) = \left(\frac{g(x)}{h(x)},\quad y\frac{x^5 - x^4 - 14x^3 - 26x^2 - 67x - 21}{x^5 - x^4 - 2x^3 + 2x^2 + x - 1} \right)$$

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Input: Finite kernel $K \subset E(k)$ of order n,

Output: Rational fractions $\phi(x,y)$;

Complexity: $\tilde{O}(n)$ operations over k.

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Complexity: $O(n)$ operations over k. Fast? YES! Optimal? NO!

One step back…

$$P(X) = \prod_{i=0}^{n-1} (X - i)$$

One step back…

$$P(\alpha) = \prod_{i=0}^{n-1} (\alpha - i)$$

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Assume $n = ab$

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$$P(\alpha) = \prod_{i=0}^{a-1} \prod_{j=0}^{b-1} (\alpha - i - j \cdot a)$$

$$G(Y) = \prod_{j=0}^{b-1} (\alpha - Y - j \cdot a)$$

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Assume $n = ab$

$$P(\alpha) = \prod_{i=0}^{a-1} G(i)$$

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Assume $n = ab$

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Pollard '74, Strassen '76: deterministic integer factorization in $O(n^{1/4})$.

Chudnovsky[2] '88: n-th term of a holonomic sequence.

Bostan '20: n-th term of a q-holonomic sequence.

Why did the trick work?

- An arithmetic decomposition of the root set: $i \mapsto i + a \cdot j$;
- Efficient to compute giant steps: $0, a, 2a, \dots, (b-1)a$;
- Only univariate polynomials.

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- Only univariate polynomials???

Wait! It works for any algebraic group! Or does it?

…and two steps forward

$$h(x(Q)) = \prod_{P\in K} (x(Q) - x(P))$$

. . . and two steps forward

$$h(x(Q)) = \prod_{i=1}^{n} (x(Q) - x([i]P))$$

…and two steps forward

$$h(x(Q)) = \prod_{i=1}^{a}\prod_{j=1}^{b}(x(Q) - x([i]P + [a\cdot j]P))$$

…and two steps forward

$$h(x(Q)) = \prod_{i=1}^{a}\prod_{j=1}^{b}(x(Q) - x([i]P + [a\cdot j]P))$$

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Biquadratic relations (Montgomery model)

Let $E : y^2 = x^3 + Ax^2 + x$, let $P, Q \in E$,

$$\begin{aligned}(X - x(P+Q))(X - x(P-Q)) = {} & X^2 \\ & - 2\frac{(x(P)x(Q)+1)(x(P)+x(Q)) + 2Ax(P)x(Q)}{(x(P)-x(Q))^2}X \\ & + \frac{(x(P)x(Q)-1)^2}{(x(P)-x(Q))^2}\end{aligned}$$

assuming $0 \notin \{P, Q, P+Q, P-Q\}$.

Back to polynomials

$$h(x(Q)) = \prod_{i=1}^{a} \prod_{j=1}^{b} (x(Q) - x([i]P + [a \cdot j]P))$$

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Back to polynomials

$$h(x(Q)) = \prod_{i\in I}\prod_{j\in J}(x(Q) - x([i]P + [j]P))(x(Q) - x([i]P - [j]P))$$

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Additional remarks

- Same technique to evaluate the isogeny numerator.
- Similar technique to compute image curve equation.
- Compatible with projective coordinates.
- Similar techniques for y-coordinate (formal derivatives).
- Also applies to kernel points defined over algebraic extensions of k (but in the worst (generic) case it provides no gain)
- Might extend to theta-functions more general than $x(\cdot)$.

Why is this important?

Every efficient isogeny based cryptosystem evaluates tons of isogenies:

SIDH (De Feo, Jao, Plût ’12): only isogenies of degree 2 and 3.

CSIDH (Castryck, Lange, Martindale, Panny, Renes ’18): degrees up to 587.

CSURF (Castryck, Decru ’20): degrees up to 389.

B-SIDH (Costello ’19): degrees in the millions!

others: Galbraith, Petit, Silva ’17,
Delpech de Saint Guilhem, Kutas, Petit, Silva ’19,
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others: Galbraith, Petit, Silva ’17,
Delpech de Saint Guilhem, Kutas, Petit, Silva ’19, needs work
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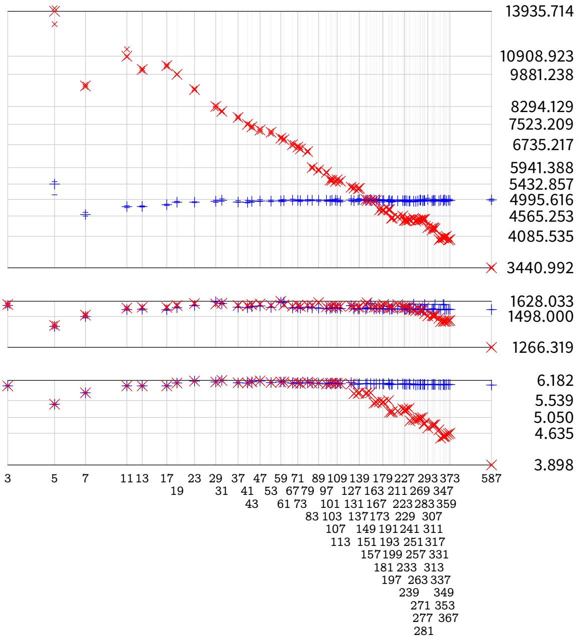


Performance of new (red) against old (blue) algorithm, in three different implementations.

Time to evaluate an isogeny with base field CSIDH-512. x-axis: isogeny degree, y-axis:

Top: Cycle counts of pure C implementation based on Flint.

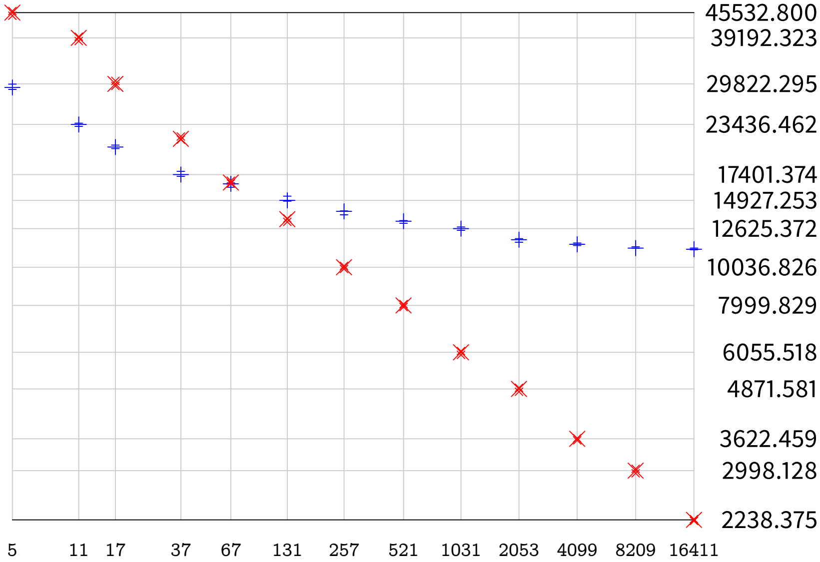
Middle: Cycle counts of assembly optimized implementation based on original CSIDH-512.

Bottom: $\mathbb{F}_p$ multiplication counts of the assembly optimized implementation.

CSIDH results

- Assembly implementation improves CSIDH-512 by 1%, CSIDH-1024 by 8%.
- Slightly smaller gain expected on CSURF-512.
- Not constant-time.
- We only applied algebraic algorithms for polynomial multiplication.
  There is still some room for improvement.

Very large degrees



Performance comparison of new (red) against old (blue) algorithm in a Julia/Nemo implementation on a 256-bits base field.

x-axis: isogeny degree.

y-axis: cycle counts.

B-SIDH results

B-SIDH: a variation on SIDH with optimal field size

- First NIST level 1 secure instantiation of B-SIDH (256 bits base field).
- High level, unoptimized implementation in Julia/Nemo.
- Largest isogeny degree: $6548911 \approx 2^{23}$.
- Alice completes one round of key exchange in 0.56s, against 2s for old algorithm.
- Bob completes one round of key exchange in 10s, against 10 minutes for old algorithm!

Implementation details

Code (5 different implementations!) at: `https://velusqrt.isogeny.org`

- Lots of exploitable symmetries due to the Montgomery form,
- Interesting challenge: fast polynomial multiplication for small degree polynomials over moderately sized fields:
  - Naive, Karatsuba,
  - Middle product algorithms.
  - Toom-Cook? For what sizes?
  - We did not explore Kronecker substitution at all!
- Scaled remainder trees to reduce number of inversions in multi-point evaluation.

Open questions

- Constant time implementation.
- Impact CSIDH? On isogeny action crypto in general?
- What about memory-constrained architectures?
- Lower bounds? See also VDFs...

https://velusqrt.isogeny.org

Thank you

https://defeo.lu/

@luca_defeo