The isogeny toolbox

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- Still the smallest keys;
- No progress on the generic isogeny problem, despite SIDH attacks;
- Very active field, fast progress;
- Credible alternative in case other pq-schemes fail;
- …but still a long way to go!

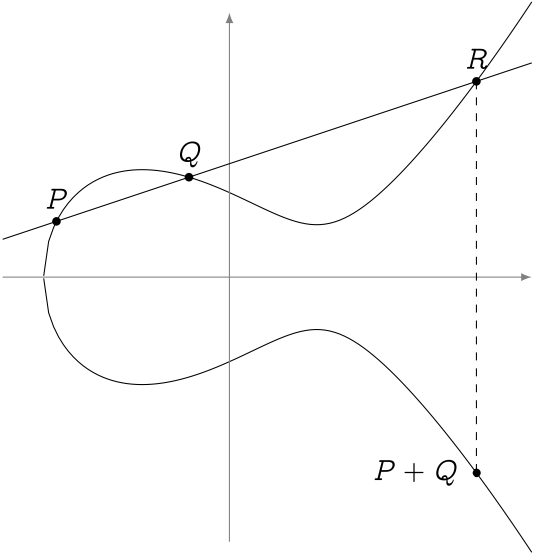
Elliptic curves

$$y^2 = x^3 + ax + b$$

Bezout's theorem

Every line cuts E in exactly three points (counted with multiplicity).

Define a group law such that any three colinear points add up to zero.

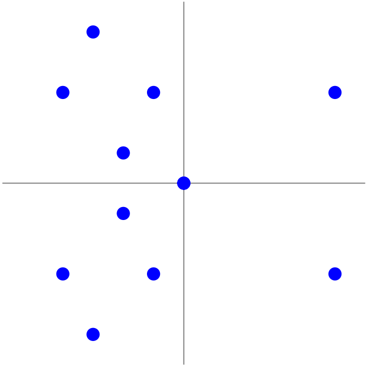


Isogenies = finite-kernel *algebraic* group morphisms: $E \to E'$

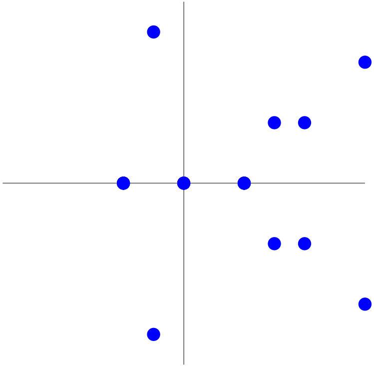
Endomorphisms = isogenies $E \to E$

Isogenies: an example over $\mathbb{F}_{11}$

$E : y^2 = x^3 + x$

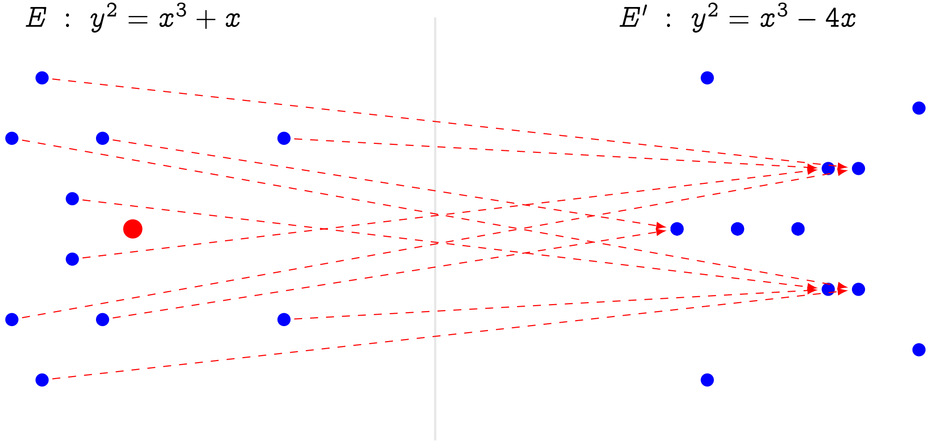


$E' : y^2 = x^3 - 4x$



$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \quad y\frac{x^2 - 1}{x^2} \right)$$

Isogenies: an example over $\mathbb{F}_{11}$



$$\phi(x, y) = \left(\frac{x^2 + 1}{x}, \quad y \frac{x^2 - 1}{x^2} \right)$$

- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Anatomy of an isogeny

$$\phi(x, y) = \left(\frac{x^4 - x^3 + 11x^2 + 9x + 12}{x^3 - x^2 - x + 1},\ y\frac{x^5 - x^4 - 14x^3 - 26x^2 - 67x - 21}{x^5 - x^4 - 2x^3 + 2x^2 + x - 1} \right)$$

Anatomy of an isogeny

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degree -1

Anatomy of an isogeny

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kernel polynomial

Anatomy of an isogeny

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Point of order 2

Two points of order 4

Anatomy of an isogeny

degree

$$\phi(x,y) = \left(\frac{x^4 - x^3 + 11x^2 + 9x + 12}{h(x)},\ y\frac{x^5 - x^4 - 14x^3 - 26x^2 - 67x - 21}{x^5 - x^4 - 2x^3 + 2x^2 + x - 1}\right)$$

$$h(x) = \prod_{P\in K\setminus\{0\}} \big(x - x(P)\big)$$

Anatomy of an isogeny

computed by Vélu–Elkies–Kohel formulas

$$\phi(x,y) = \left(\frac{g(x)}{h(x)}, \quad y\frac{x^5 - x^4 - 14x^3 - 26x^2 - 67x - 21}{x^5 - x^4 - 2x^3 + 2x^2 + x - 1} \right)$$

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Input: Finite kernel $K \subset E$ of order d;

Output: Rational fractions $\phi(x,y)$;

Complexity: $\tilde{O}(d)$ operations.

How many isogenies?

Finite subgroups of order d

$K \subset E$

$\longleftrightarrow$

Isogenies of degree d

$\phi : E \to E/K$

(up to composing with isomorphism)

How many isogenies?

Finite subgroups of order d ⟷ Isogenies of degree d

$$K \subset E$$

$$\phi : E \to E/K$$

(up to composing with isomorphism)

Examples:

If d is prime $\to$ at most $d+1$ possible kernels,

In general $\to$ at most $\approx d$ possible kernels.

The isogeny problem

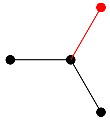
E E'

The isogeny problem

$$E \xrightarrow{\quad ?? \quad} E'$$

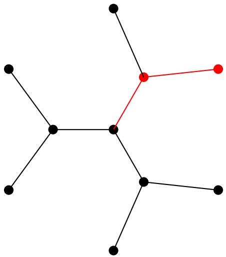
Isogeny graphs

$$\frac{x^2 + \cdots}{x + \cdots}$$



Isogeny graphs

$$\frac{x^2 + \cdots}{x + \cdots} \circ \frac{x^2 + \cdots}{x + \cdots}$$



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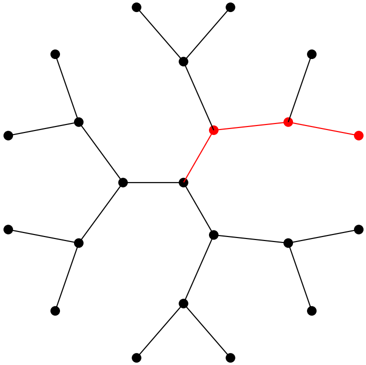

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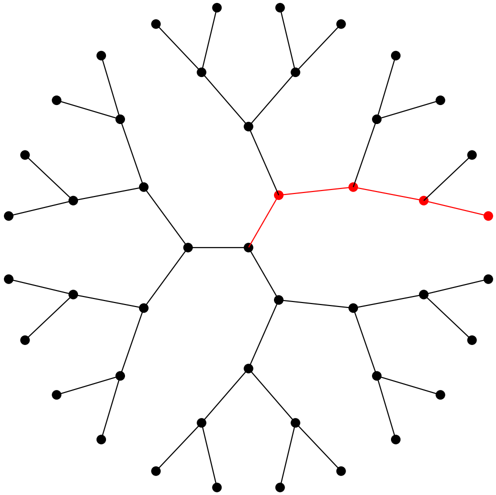

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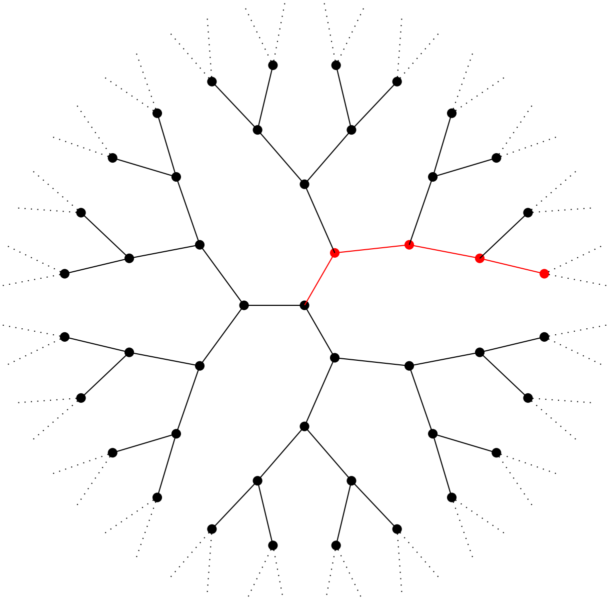
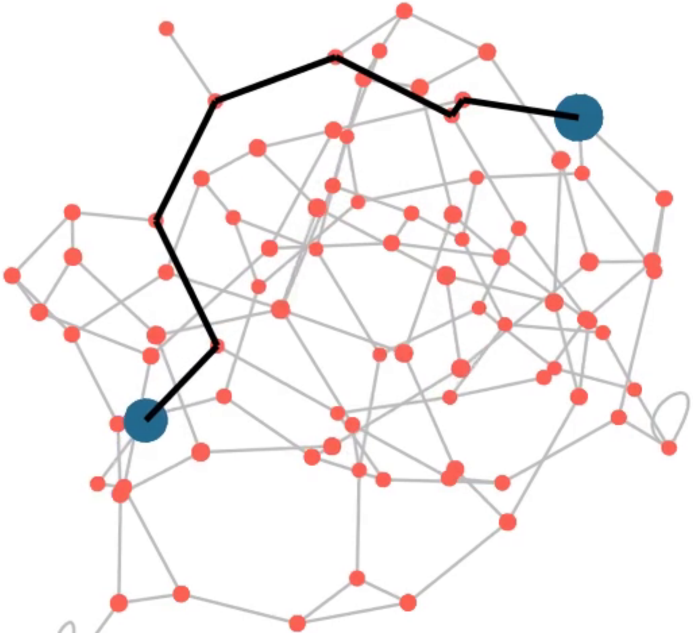




The *smooth* criminals

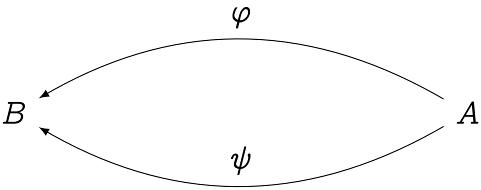
2006 Charles-Goren-Lauter hash function

2006 Couveignes-Rostovtsev-Stolbunov key exchange

2011 SIDH key exchange

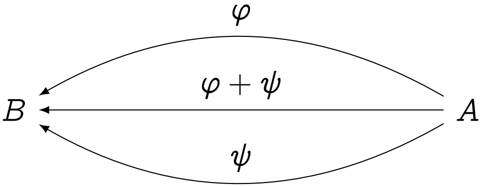
2018 CSIDH key exchange

Groups of isogenies



$$\varphi, \psi \in \mathrm{Hom}(A, B)$$

Groups of isogenies



$$\varphi, \psi \in \mathrm{Hom}(A, B)$$

$$(\varphi + \psi)(P) := \varphi(P) + \psi(P)$$

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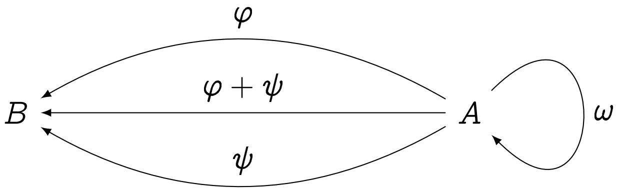


$$\varphi, \psi \in \mathrm{Hom}(A, B) \qquad\qquad \omega \in \mathrm{Hom}(A, A)$$

$$(\varphi + \psi)(P) := \varphi(P) + \psi(P)$$

Endomorphism rings

- $\mathrm{Hom}(A, B)$ is a group

- Distributivity:

$$\varphi \circ (\psi + \chi) = (\varphi \circ \psi) + (\varphi \circ \chi)$$

$$(\psi + \chi) \circ \varphi = (\psi \circ \varphi) + (\chi \circ \varphi)$$

- It follows that $\mathrm{End}(A) := \mathrm{Hom}(A, A)$ is a ring.

Endomorphism rings

$\mathrm{End}(E)$ is a free $\mathbb{Z}$-module of rank 1, 2 or 4. As a ring:

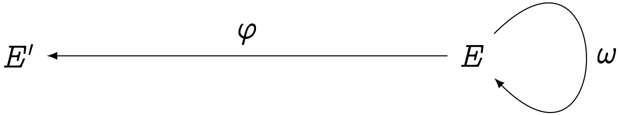
1) $\mathrm{End}(E) \simeq \mathbb{Z}$;
2) $\mathrm{End}(E) \subset$ quadratic imaginary field;
4) $\mathrm{End}(E) \subset$ quaternion algebra.

Isogenies = Ideals

$$E' \xleftarrow{\quad\varphi\quad} E$$

$$\varphi \in \mathrm{Hom}(E, E')$$

Isogenies = Ideals

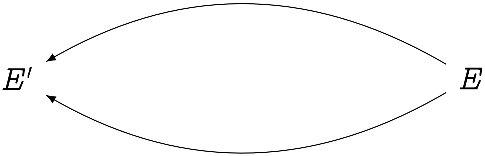


$$\varphi \circ \omega \in \mathrm{Hom}(E, E')$$

Isogenies = Ideals

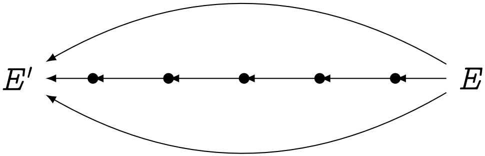
$$\omega' \circ \varphi \in \mathrm{Hom}(E, E')$$

Smoothifying ideals



$$\mathrm{Hom}(E, E') = \mathbb{Z}\varphi_1 + \mathbb{Z}\varphi_2$$

Smoothifying ideals



$$\mathrm{Hom}(E, E') = \mathbb{Z}\varphi_1 + \mathbb{Z}\varphi_2$$

$$\deg(a\varphi_1 + b\varphi_2) = \textit{smooth}$$

Smoothifying in quadratic imaginary rings

Algorithm: Index calculus

Cost: (sub)exponential complexity, fast in practice

Used in: CSI-FiSh signature scheme (2019)

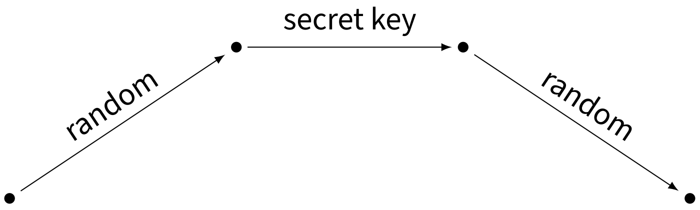
Smoothifying in quaternion rings

Algorithm: Kohel–Petit–Tignol–Lauter (KLPT)

Cost: polynomial time

Used in: Galbraith–Petit–Silva (2016) and SQIsign (2020) signatures

Basically every isogeny signature



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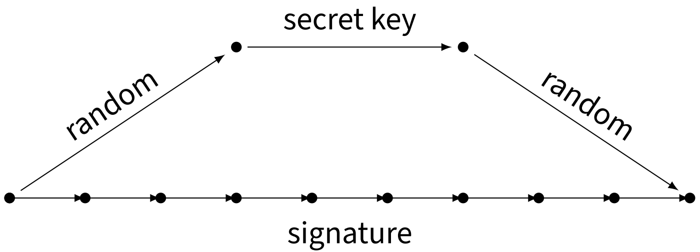


SQIsign

	Bytes		Mcycles			
Secret Key	Public Key	Signature	Keygen	Sign	Verify	Security
782	64	177	3,728	5,779	108	NIST-1
1,138	96	263	23,734	43,760	654	NIST-3
1,509	128	335	91,049	158,544	2,177	NIST-5

What does it mean to “compute” an isogeny?

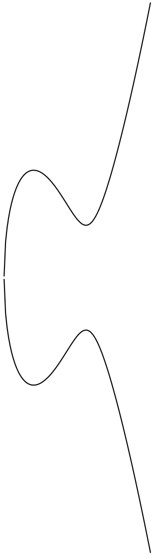
Definition (Isogeny representation)

A *representation* of an isogeny $\varphi : E \to E'$ is an algorithm / Turing machine / arithmetic circuit that:

- on input $P \in E$
- outputs $\varphi(P) \in E'$

in time polynomial in $\log(\deg \varphi)$.

Higher dimensional abelian varieties



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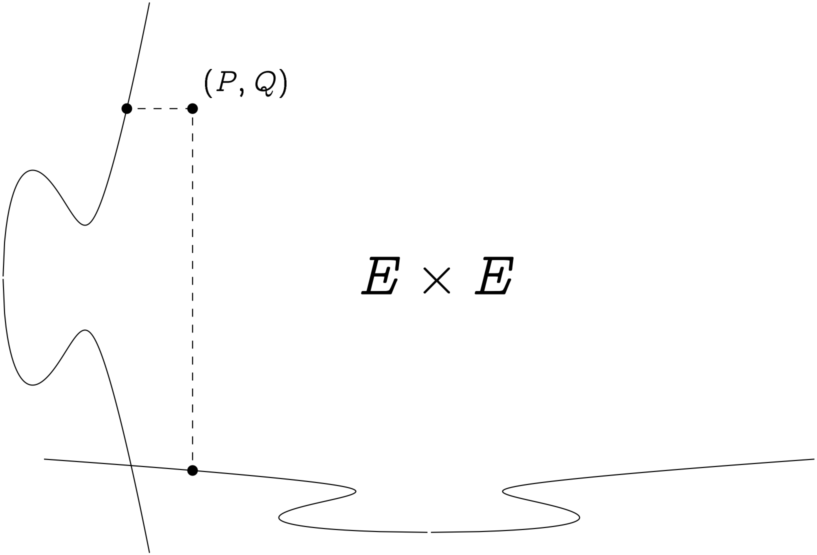
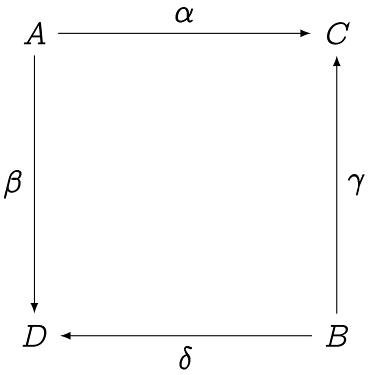


Higher dimensional isogenies



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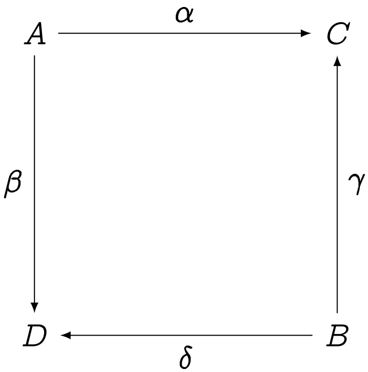
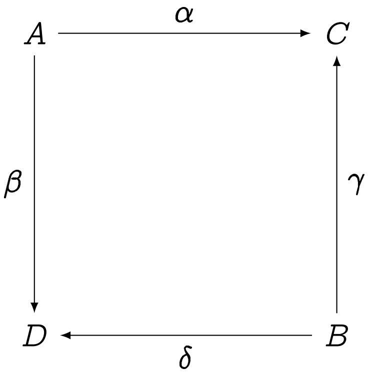


$$A \times B \longrightarrow C \times D$$

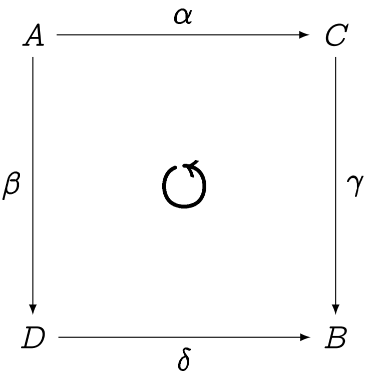
$$(P, Q) \longmapsto \big(\alpha(P) + \gamma(Q), \beta(P) + \delta(Q)\big)$$

Higher dimensional isogenies



$$A \times B \longrightarrow C \times D$$
$$(P, Q) \longmapsto \big(\alpha(P) + \gamma(Q), \beta(P) + \delta(Q)\big)$$
$$= \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}$$

Kani's lemma

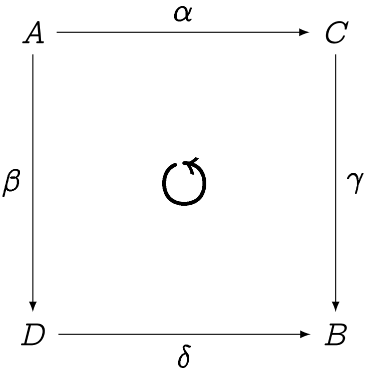


Let $\deg\alpha = \deg\delta$ and $\deg\beta = \deg\gamma$ coprime. The isogeny defined by

$$\Phi : A \times B \longrightarrow C \times D$$
$$\begin{pmatrix} P \\ Q \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & \tilde{\gamma} \\ -\beta & \tilde{\delta} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}$$

is a $\big(\deg(\alpha) + \deg(\beta)\big)$-isogeny if and only if the diagram commutes.

Kani's lemma



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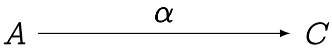
Note: $\Phi(P, 0) = \big(\alpha(P), -\beta(P)\big)$.

Application: breaking SIDH



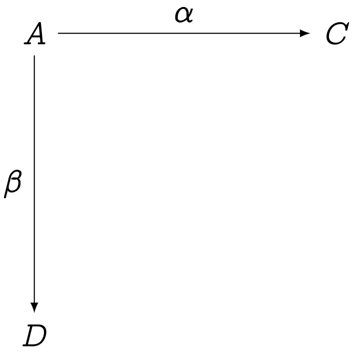
Secret: $\alpha : A \to B$ of degree $\deg(\alpha) = d$

Input: $\alpha(P)$ for all P of order $2^n > d$

Goal: Compute a representation of α

Attack (sketch):

Application: breaking SIDH



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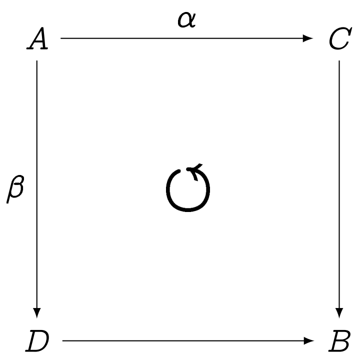

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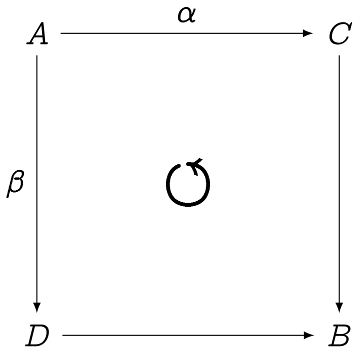

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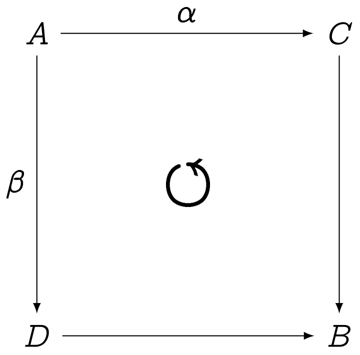

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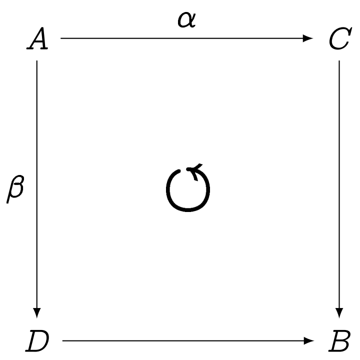

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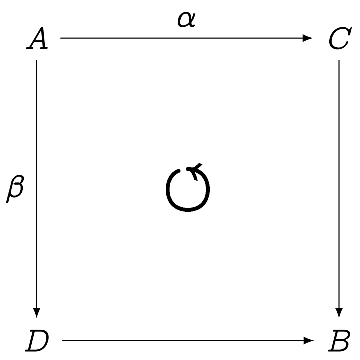

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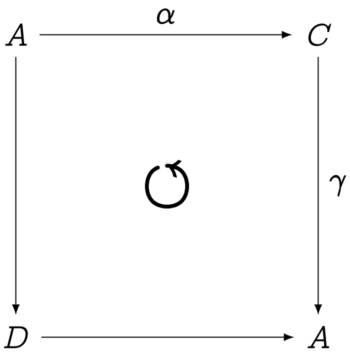
4. Then $\Phi(P', 0) = \big(\alpha(P'), \dots\big)$ for any $P' \in A$;
5. Deduce kernel of α;
6. Claim 50 K$.

Application: from quaternions to random isogenies (QFESTA)

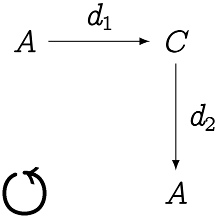


Input: $\mathrm{End}(A)$, a degree d,

Output: A random d-isogeny $A \to ?$.

1. Find endomorphism ω of degree $\deg \omega = d(2^n - d)$,
2. Factor $\omega = \alpha \circ \gamma$ with $\deg(\alpha) = d$,
3. Kani's isogeny is a representation of α.

Application: evaluate (almost) any ideal (Clapoti)

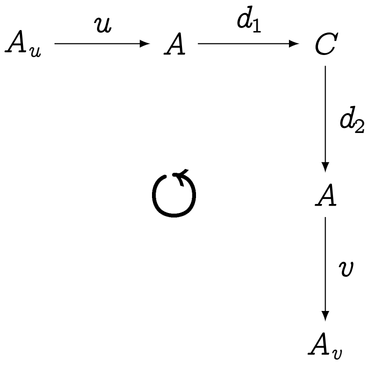


Input: $\mathrm{End}(A)$, an ideal of $\mathrm{End}(A)$,

Output: The corresponding isogeny.

1. Find random equivalent ideals of coprime degrees d_1, d_2;

Application: evaluate (almost) any ideal (Clapoti)

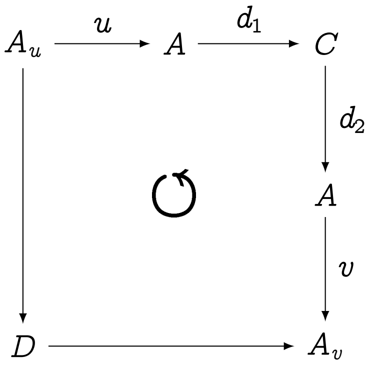


Input: $\mathrm{End}(A)$, an ideal of $\mathrm{End}(A)$,

Output: The corresponding isogeny.

1. Find random equivalent ideals of coprime degrees d_1, d_2;
2. Find integers u, v s.t. $ud_1 + vd_2 = 2^n$;
3. Find random u- and v-isogeny;

Application: evaluate (almost) any ideal (Clapoti)



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1. Find random equivalent ideals of coprime degrees d_1, d_2;
2. Find integers u, v s.t. $ud_1 + vd_2 = 2^n$;
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4. Construct Kani square.

Putting it all together: SQIsign2D-West

Bytes		Mcycles			
Public Key	Signature	Keygen	Sign	Verify	Security
66	148	60	160	9	NIST-1
98	222	170	460	29	NIST-3
130	294	360	940	62	NIST-5

Thank you

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