SQIsign
past, present and future

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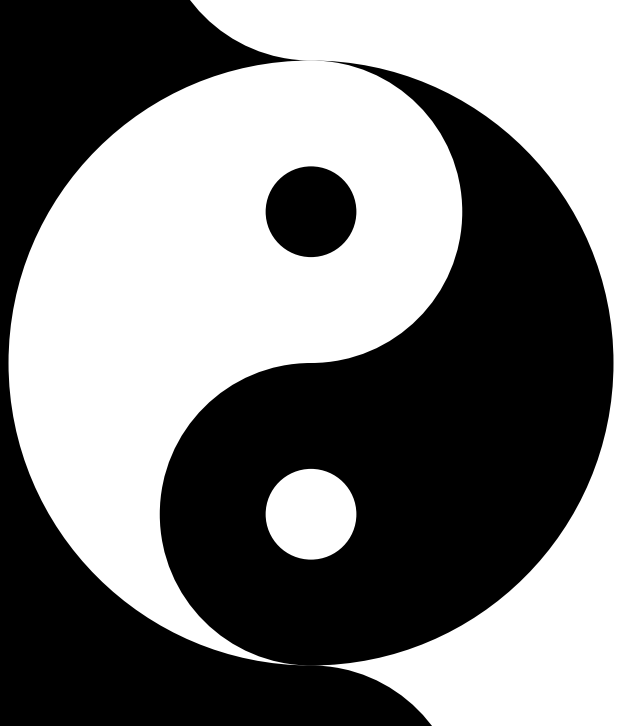
Indocrypt, Chennai, India

CRYPTANALYSIS



CRYPTOGRAPHY

2006 – CRS key exchange based on the CM group action

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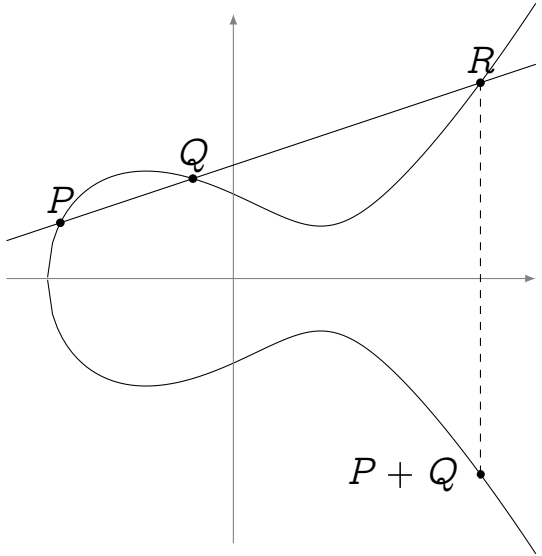
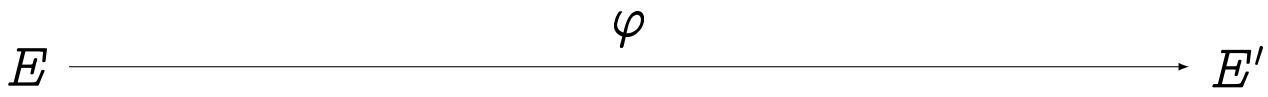
2024 – SQIsign2D

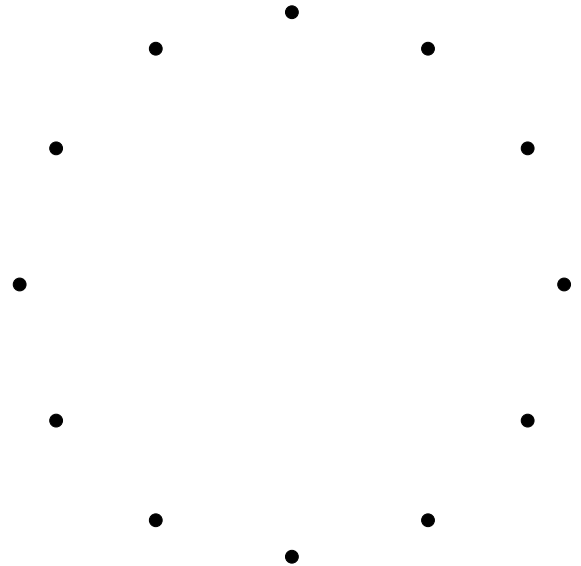


Isogenies = finite-kernel *algebraic* group morphisms of elliptic curves

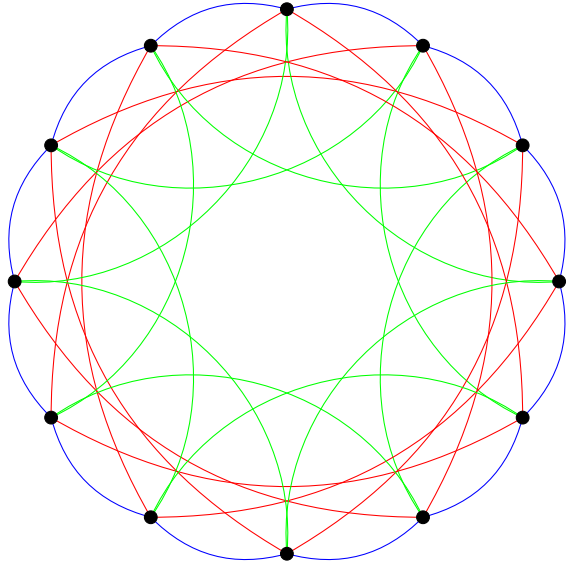
Endomorphisms = isogenies $E \to E$


E
φ
E'

Isogenous



Isogeny class



Isogeny class

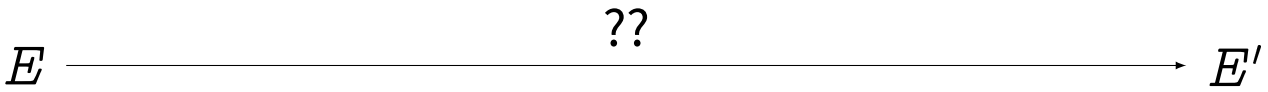
Isogeny graph

The isogeny problem

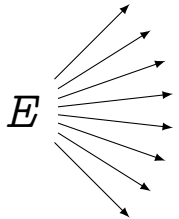
E

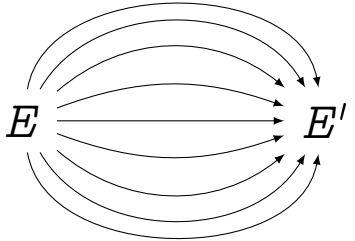
E'

The isogeny problem



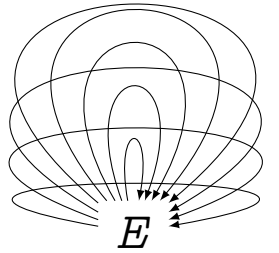
Isogeny Yantras





$\mathrm{Hom}(E, E')$

Ideal class

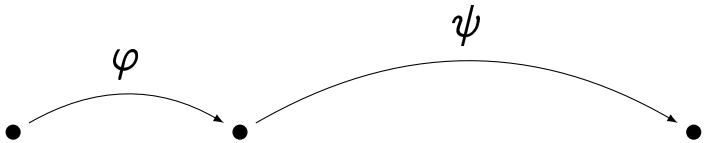


$\mathrm{End}(E)$

Order

Degree

$$\deg(\varphi) \in \mathbb{N}^+$$

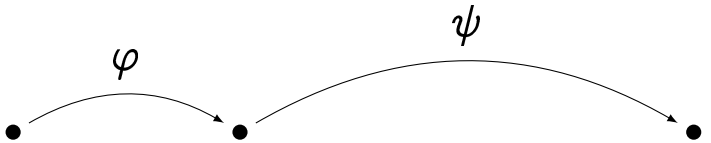


- A measure of “size”;

Degree

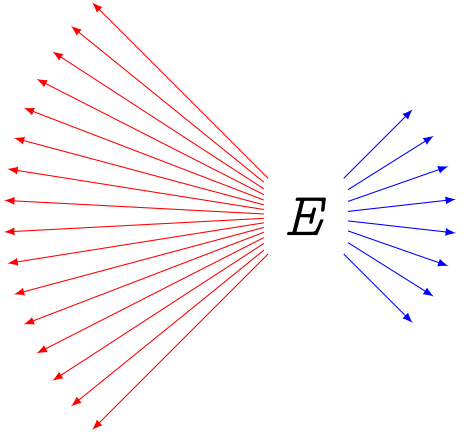
$$\deg(\varphi) \in \mathbb{N}^+$$

- A measure of “size”;
- Multiplicative;



$$\deg(\psi \circ \varphi) = \deg(\psi)\deg(\varphi)$$

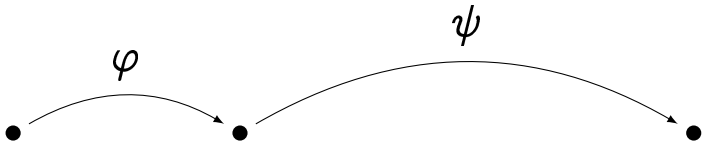
The larger the degree, the more isogenies



Degree

$$\deg(\varphi) \in \mathbb{N}^+$$

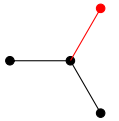
- A measure of “size”;
- Multiplicative;
- A measure of “complexity”:
  only isogenies of small degree fit into a computer!



$$\deg(\psi \circ \varphi) = \deg(\psi)\deg(\varphi)$$

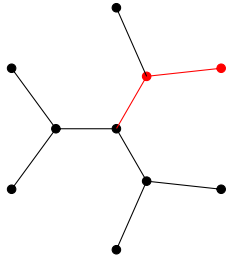
Isogenies of smooth degree

degree 2



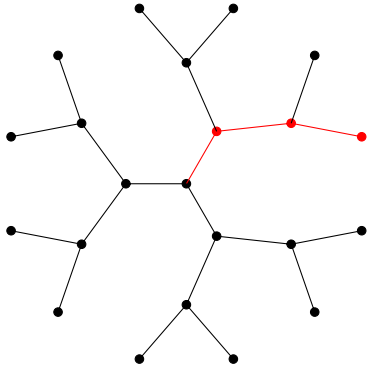
Isogenies of smooth degree

degree 4



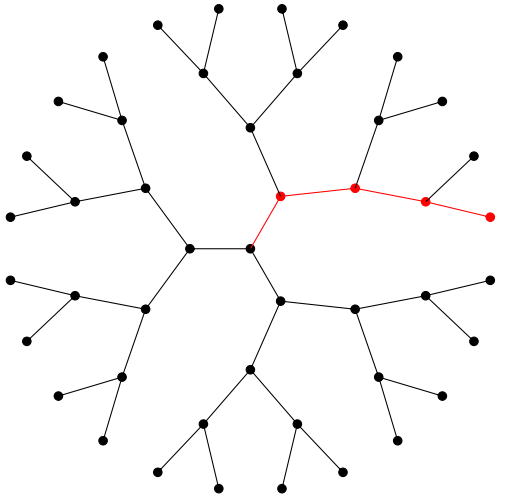
Isogenies of smooth degree

degree 8



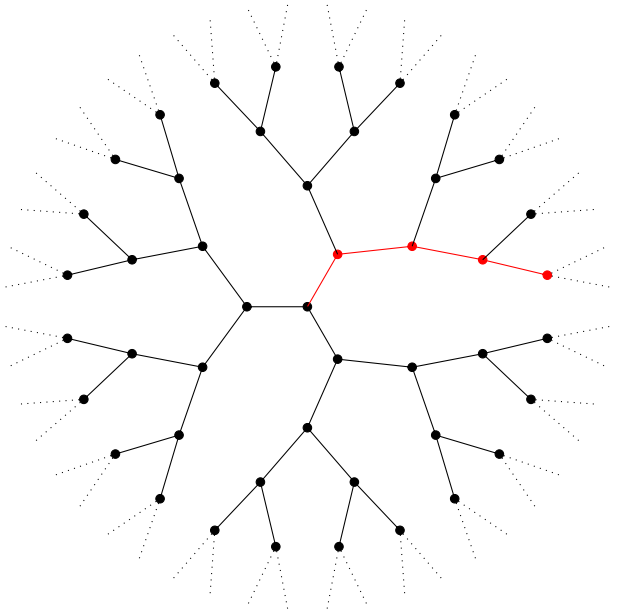
Isogenies of smooth degree

degree 16

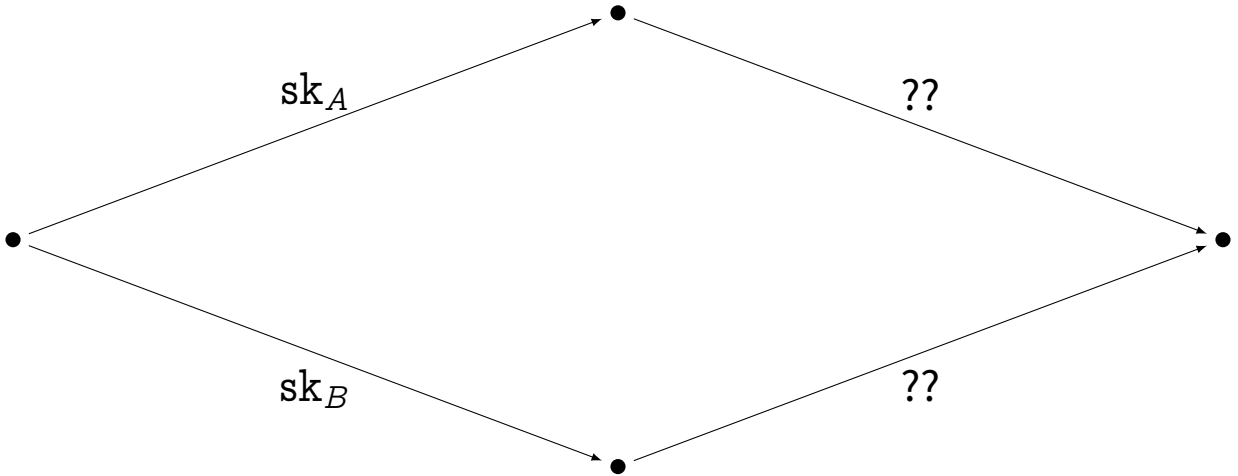


Isogenies of smooth degree

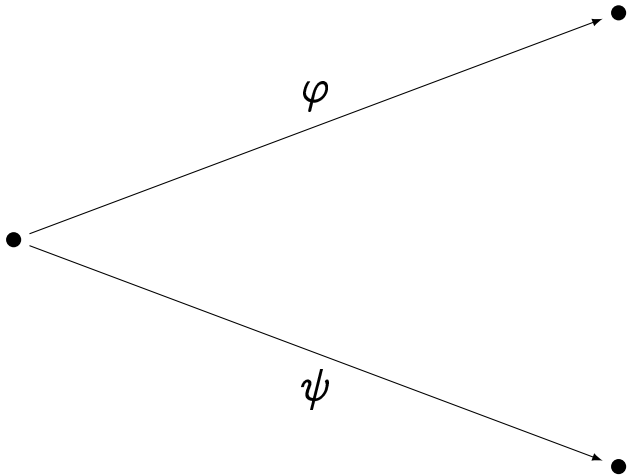
degree 2^x



Key exchange?

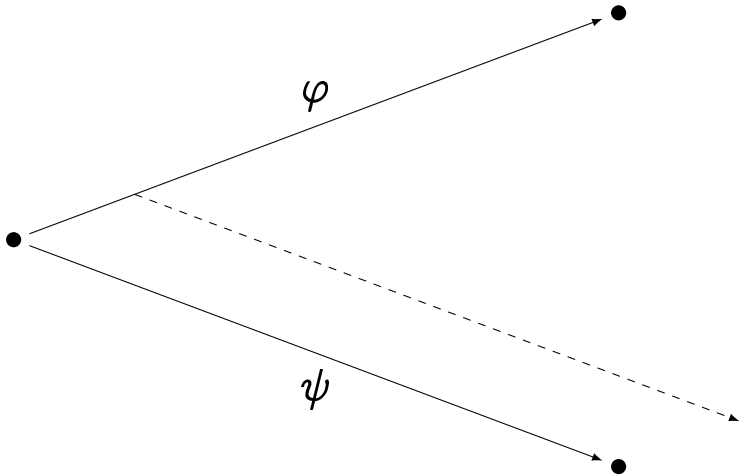


Push-forward



$$\big(\deg(\varphi), \deg(\psi)\big) = 1$$

Push-forward



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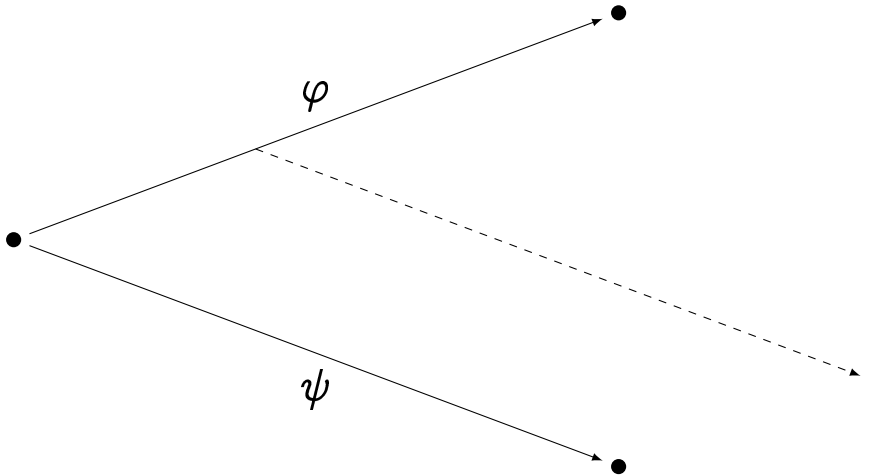

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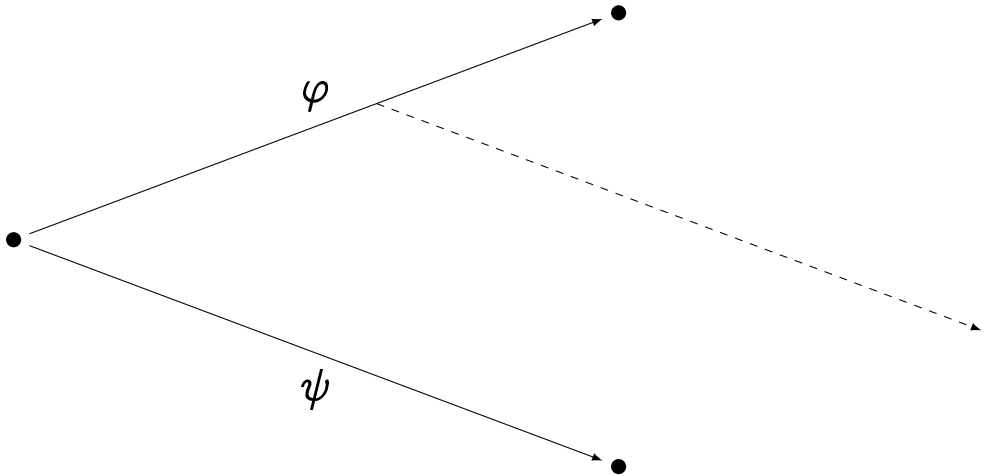

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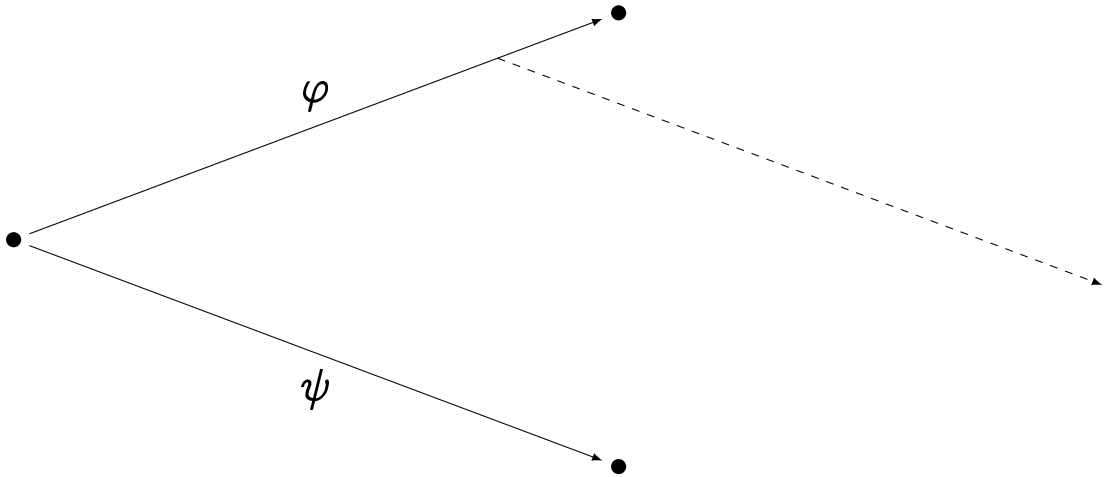

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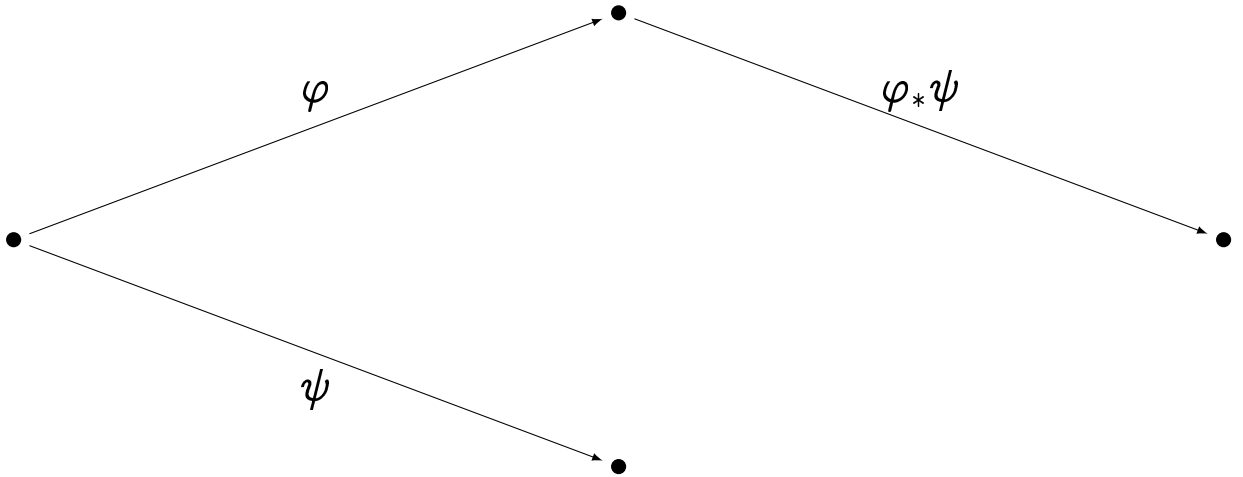

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$$\deg(\varphi_*\psi) = \deg(\psi)$$

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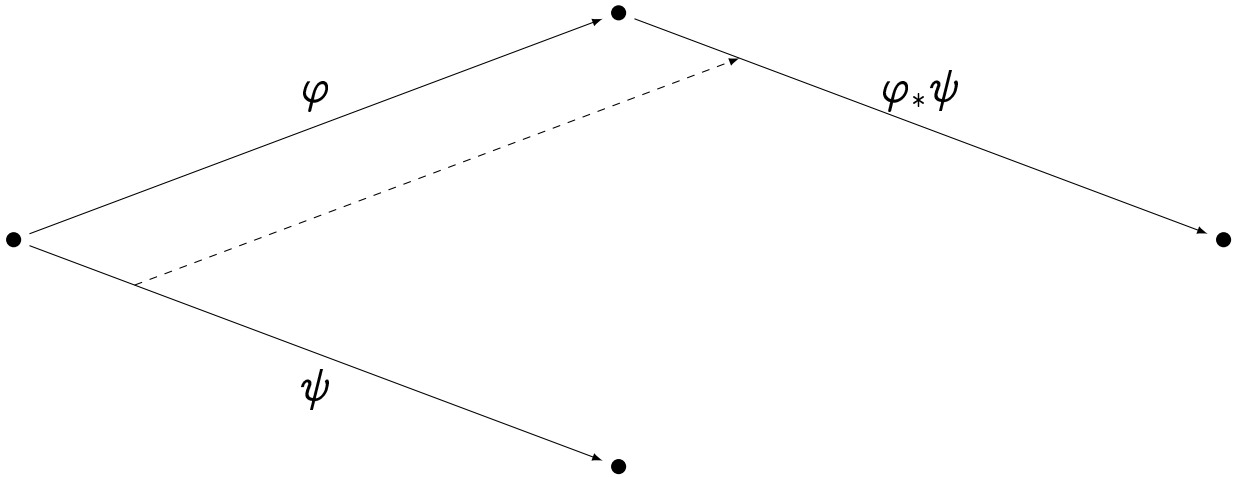

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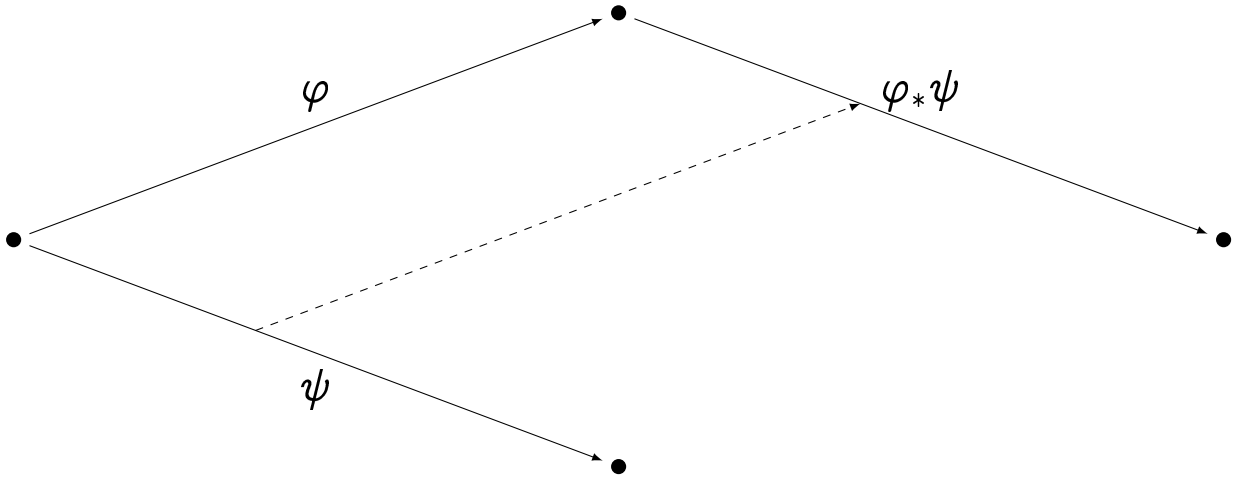

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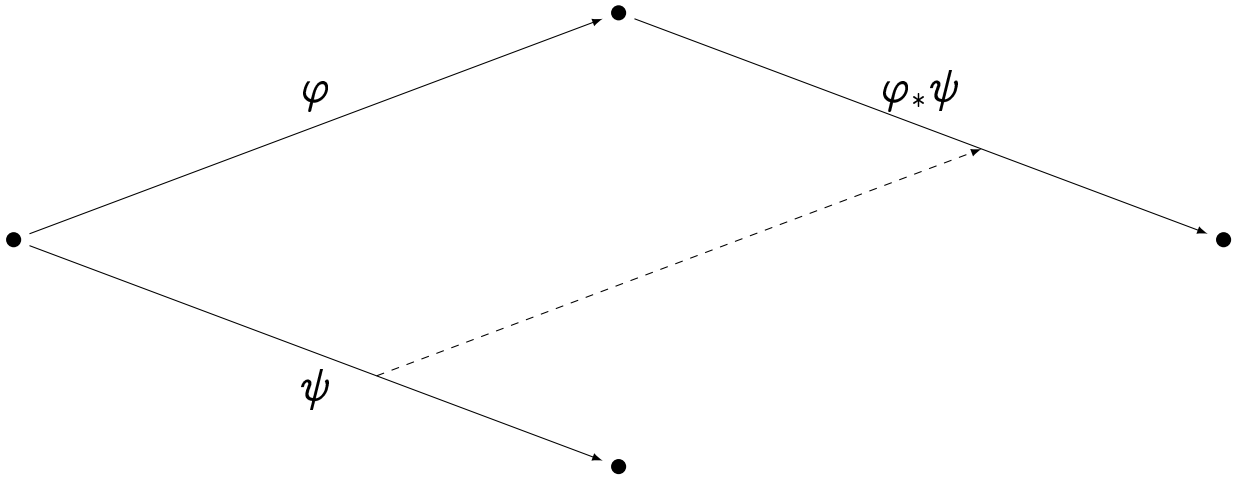

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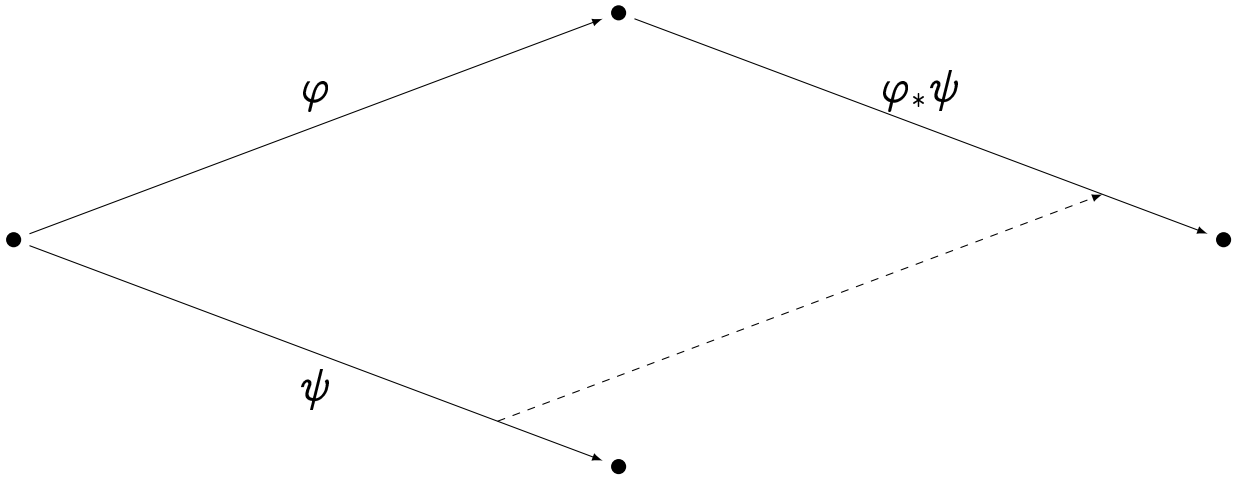

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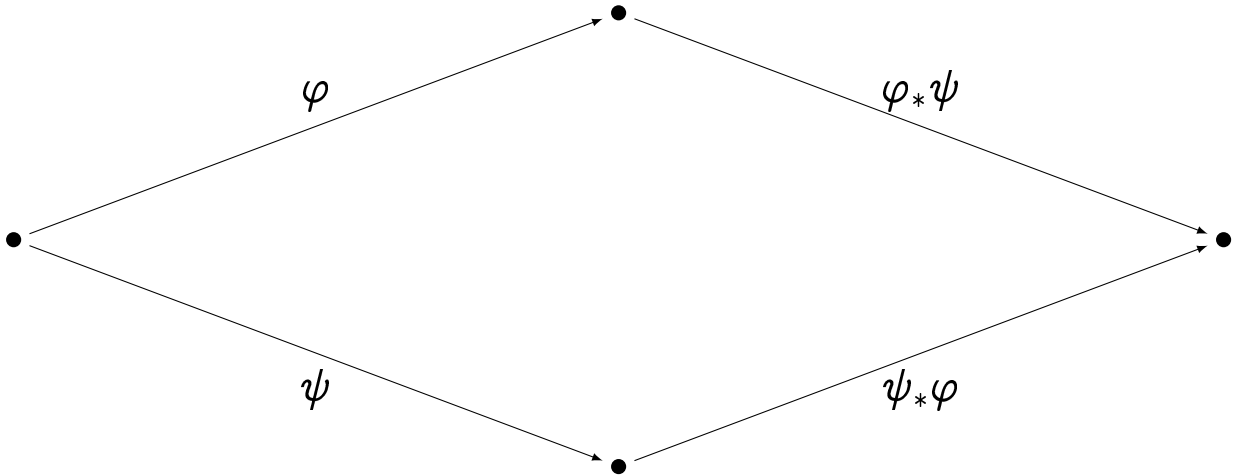
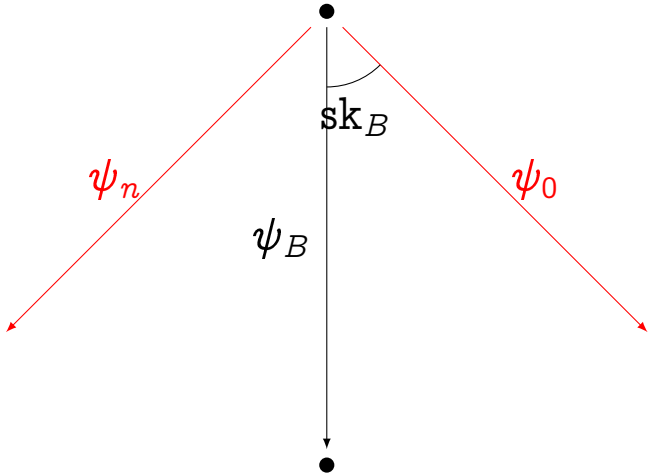


$$\big(\deg(\varphi), \deg(\psi)\big) = 1$$

$$\deg(\varphi_*\psi) = \deg(\psi)$$

$$\deg(\psi_*\varphi) = \deg(\varphi)$$

Supersingular Isogeny Diffie-Hellman (SIDH)



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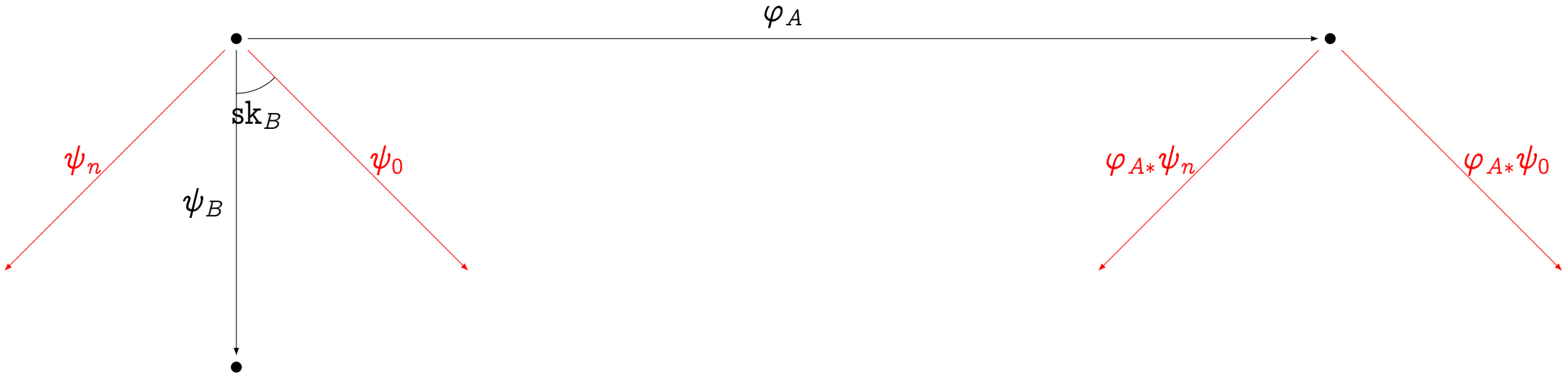

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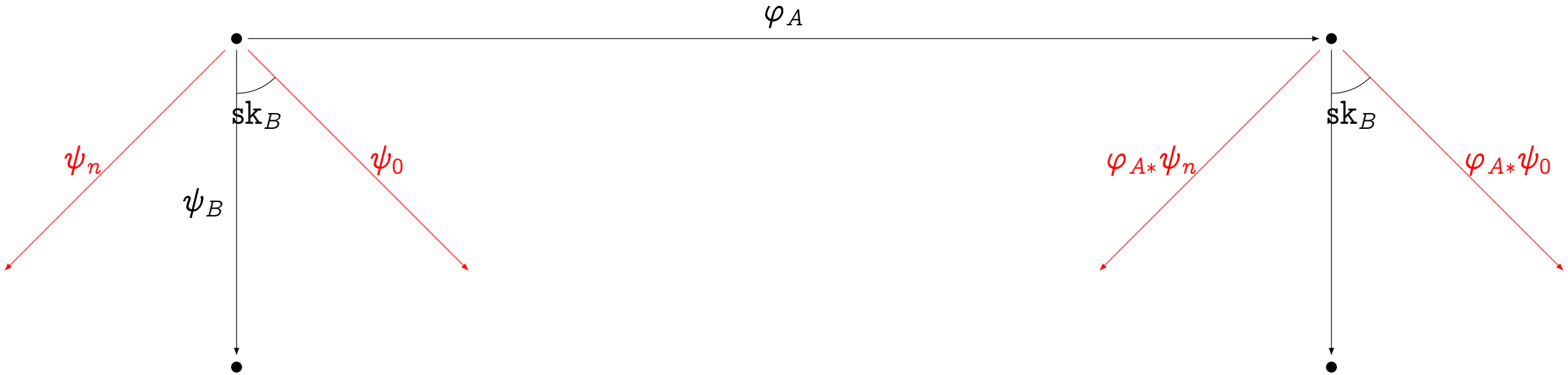

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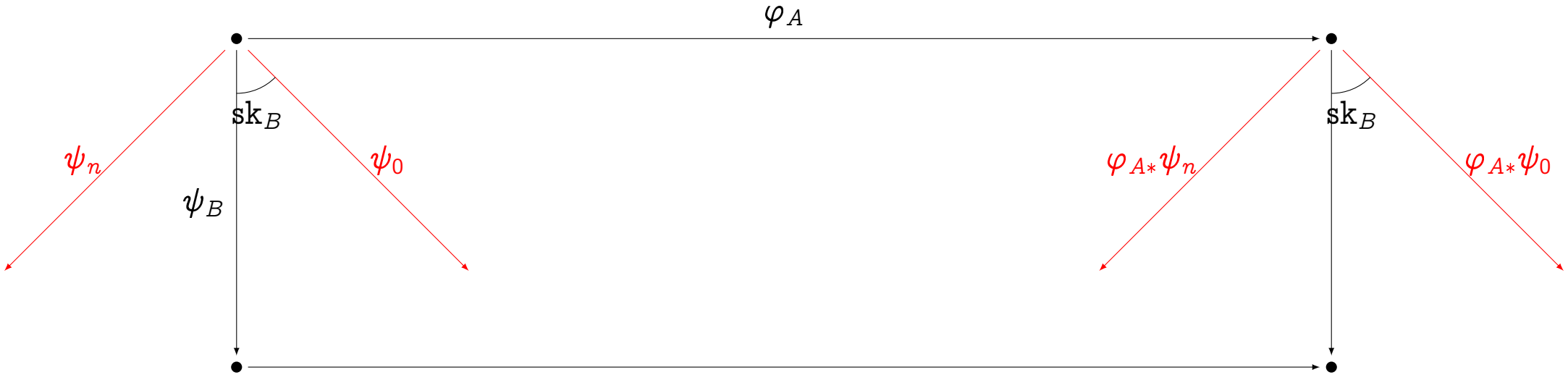
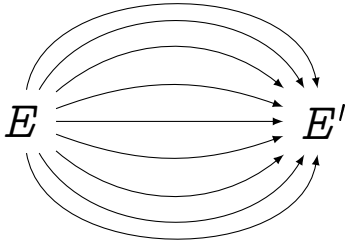
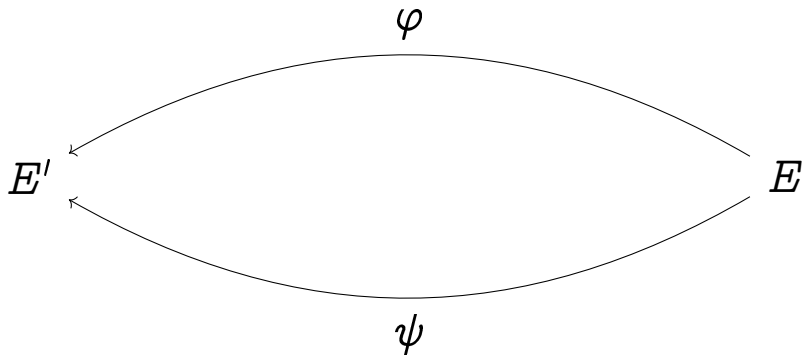


The Deuring Correspondence



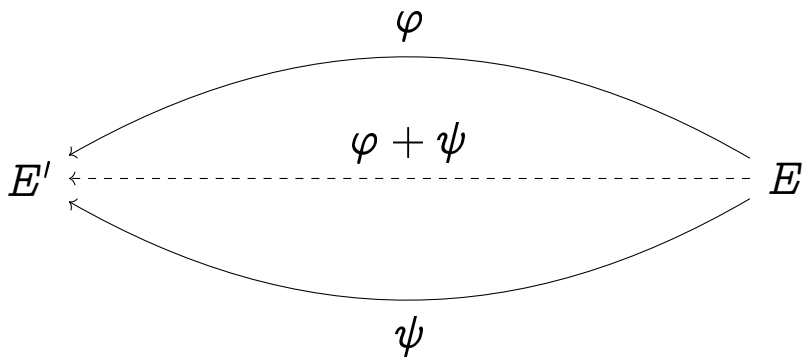
$$\mathrm{Hom}(E, E')$$

Homs are groups



$$\varphi, \psi \in \mathrm{Hom}(E, E')$$

Homs are groups



$$\varphi, \psi \in \mathrm{Hom}(E, E')$$

$$(\varphi + \psi)(P) := \varphi(P) + \psi(P)$$

Ends are rings

Distributivity

$$\varphi \circ (\psi + \chi) = (\varphi \circ \psi) + (\varphi \circ \chi)$$

$$(\psi + \chi) \circ \varphi = (\psi \circ \varphi) + (\chi \circ \varphi)$$

Supersingular endomorphism rings

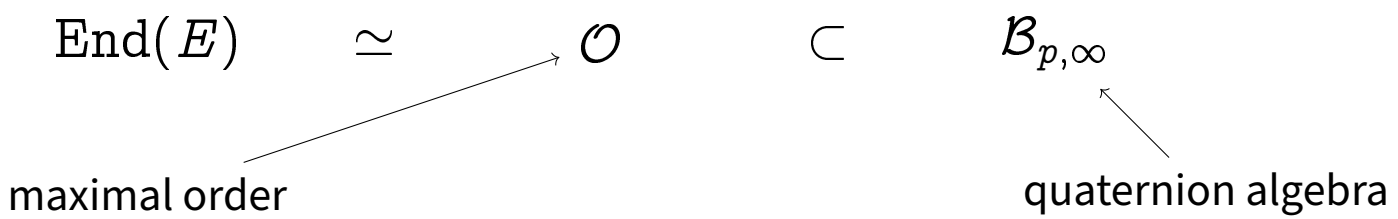
$$\mathrm{End}(E) \quad \simeq \quad \mathcal{O} \quad \subset \quad \mathcal{B}_{p,\infty}$$

Supersingular endomorphism rings

$$\mathrm{End}(E) \quad \simeq \quad \mathcal{O} \quad \subset \quad \mathcal{B}_{p,\infty}$$

quaternion algebra

Supersingular endomorphism rings

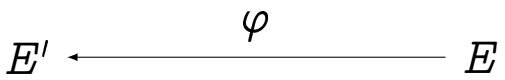


Supersingular endomorphism rings

Supersingular endomorphism rings

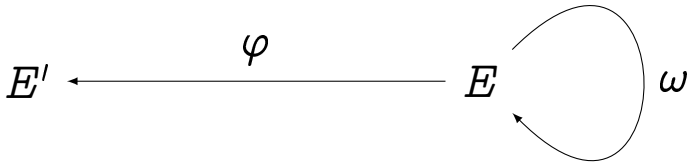
EndRing problem: given E supersingular, compute an order $\mathcal{O} \simeq \mathrm{End}(E)$.

Homs $\longleftrightarrow$ Modules $\longleftrightarrow$ Ideals



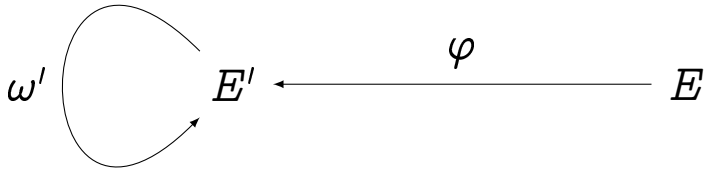
$$\varphi \in \mathrm{Hom}(E, E')$$

Homs $\longleftrightarrow$ Modules $\longleftrightarrow$ Ideals



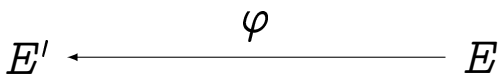
$$\varphi \circ \omega \in \mathrm{Hom}(E, E')$$

Homs $\longleftrightarrow$ Modules $\longleftrightarrow$ Ideals

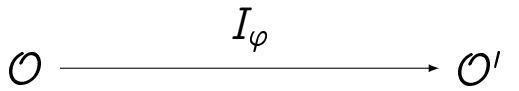


$$\omega' \circ \varphi \in \mathrm{Hom}(E, E')$$

Homs $\longleftrightarrow$ Modules $\longleftrightarrow$ Ideals



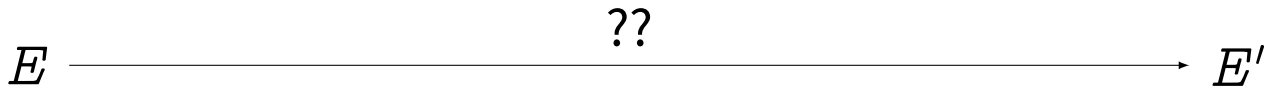
$$\varphi \in \mathrm{Hom}(E, E')$$



$$\mathrm{Hom}(E, E') \simeq I_\varphi$$
$$\mathcal{O} \supset I_\varphi \subset \mathcal{O}'$$

Endomorphism ring	Maximal order
Isogeny	Ideal
$\mathrm{Hom}(E, E')$	Ideal class
Degree	Norm

Kohel-Lauter-Petit-Tignol (KLPT)



Kohel-Lauter-Petit-Tignol (KLPT)

$$\mathrm{End}(E) \xrightarrow{\quad ?? \quad} \mathrm{End}(E')$$

Kohel-Lauter-Petit-Tignol (KLPT)

$$\mathrm{End}(E) \xrightarrow{\qquad I \qquad} \mathrm{End}(E')$$

$$N(I) = 2^x$$

Kohel-Lauter-Petit-Tignol (KLPT)

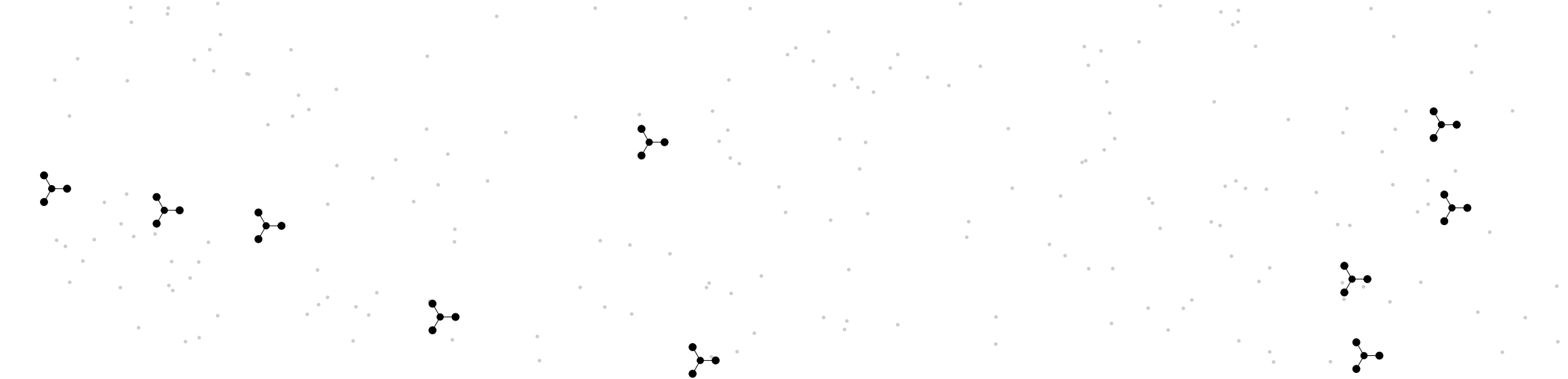
$$\mathrm{End}(E) \xrightarrow{\qquad I \qquad} \mathrm{End}(E')$$

$$N(I) = 2^x$$

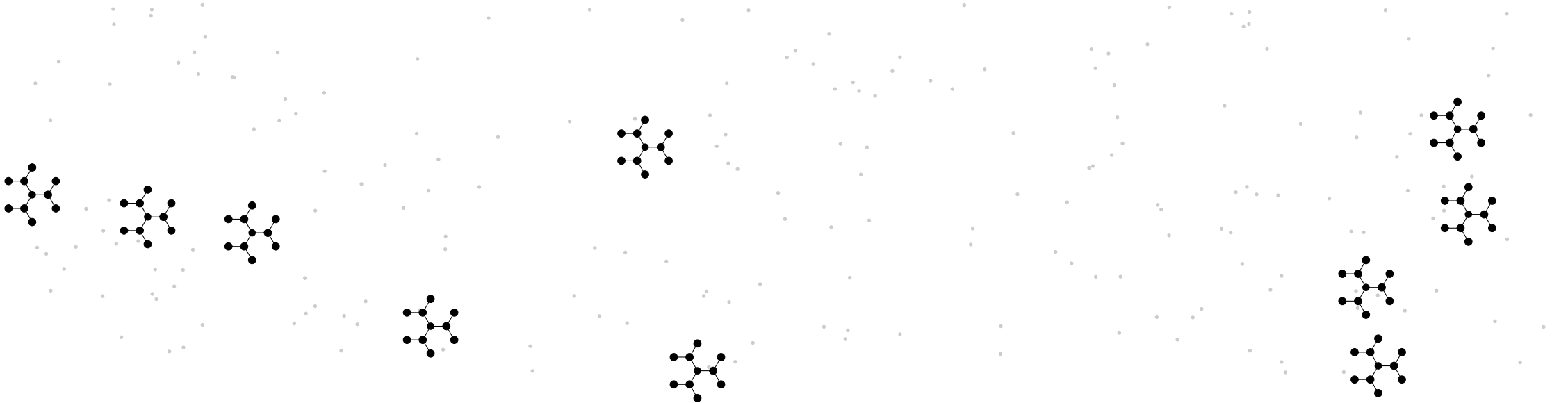
Corollary: Isogeny problem $\simeq$ EndRing problem

Contagious knowledge

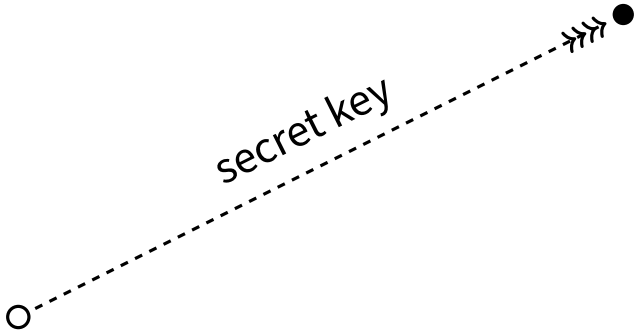
Contagious knowledge



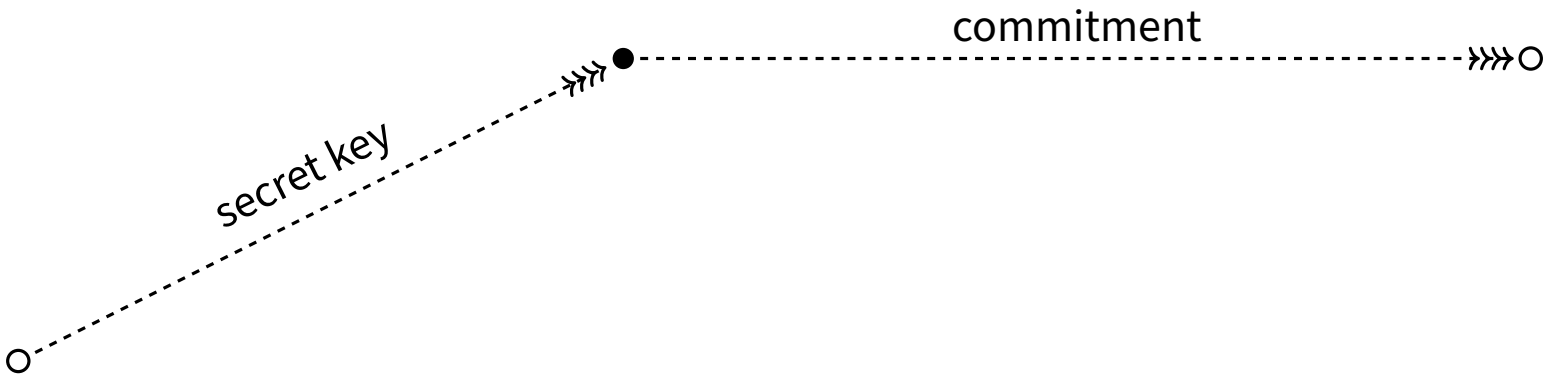
Contagious knowledge



SQIsign (identification scheme)



SQIsign (identification scheme)



SQIsign (identification scheme)

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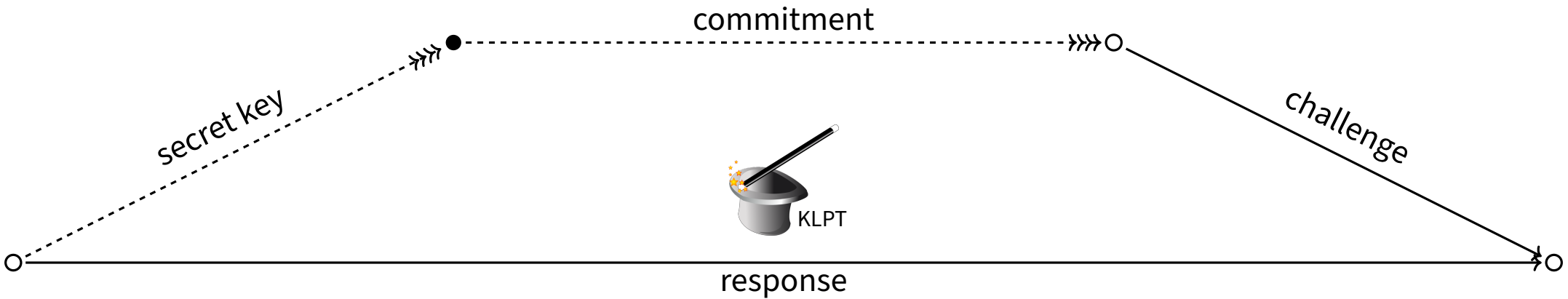
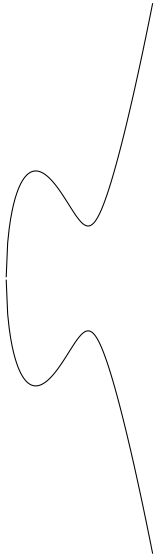


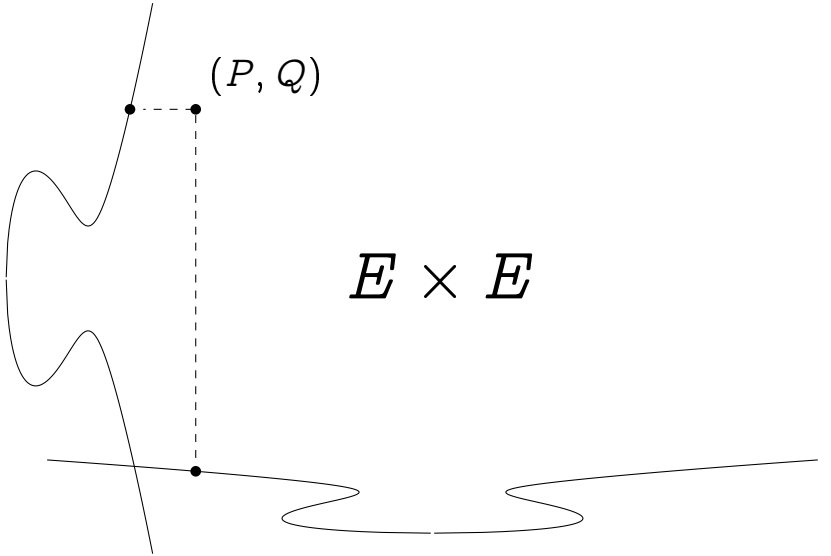
SQIsign

Bytes		Mcycles			
Public Key	Signature	Keygen	Sign	Verify	Security
64	177	3,728	5,779	108	NIST-1
96	263	23,734	43,760	654	NIST-3
128	335	91,049	158,544	2,177	NIST-5

Higher dimensional abelian varieties



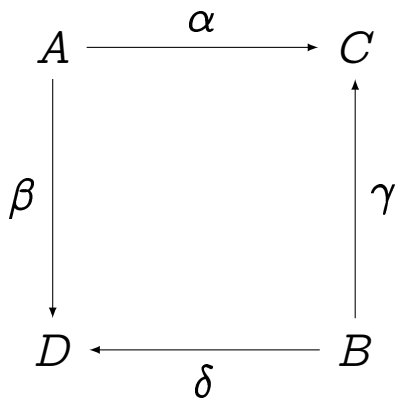
Higher dimensional abelian varieties



Higher dimensional isogenies

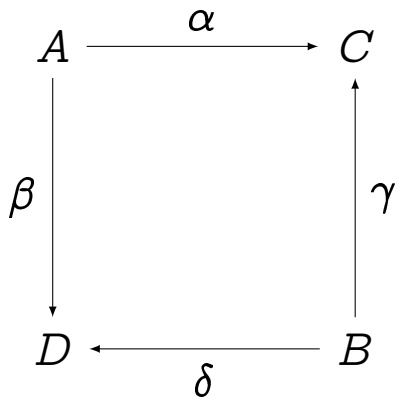
$$A \times B \longrightarrow C \times D$$

Higher dimensional isogenies



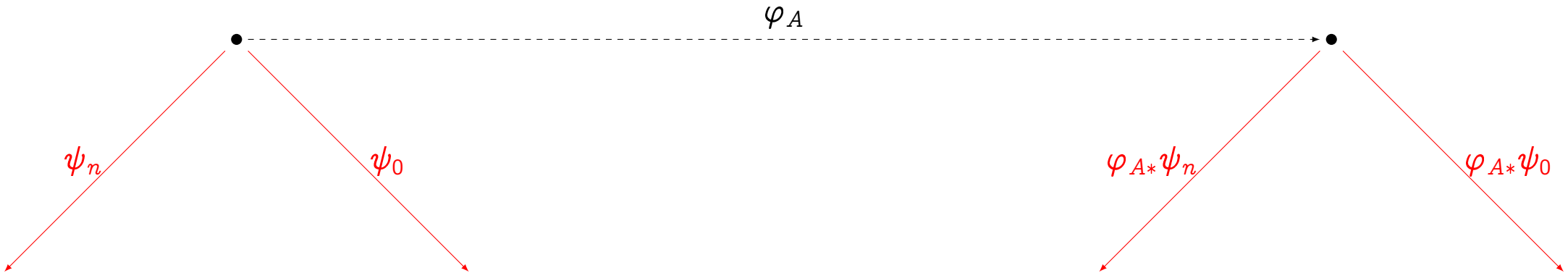
$$A \times B \longrightarrow C \times D$$
$$(P, Q) \longmapsto \big(\alpha(P) + \gamma(Q), \beta(P) + \delta(Q)\big)$$

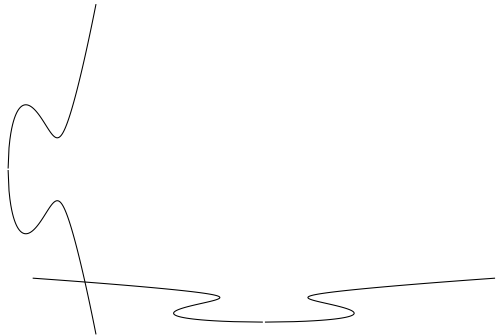
Higher dimensional isogenies



$$A \times B \longrightarrow C \times D$$
$$(P, Q) \longmapsto \big(\alpha(P) + \gamma(Q), \beta(P) + \delta(Q)\big)$$
$$= \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}$$

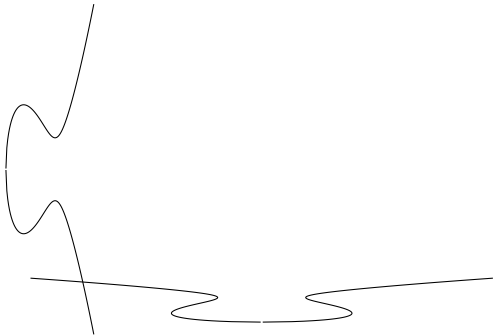
Isogeny problem + Torsion point information (SIDH)





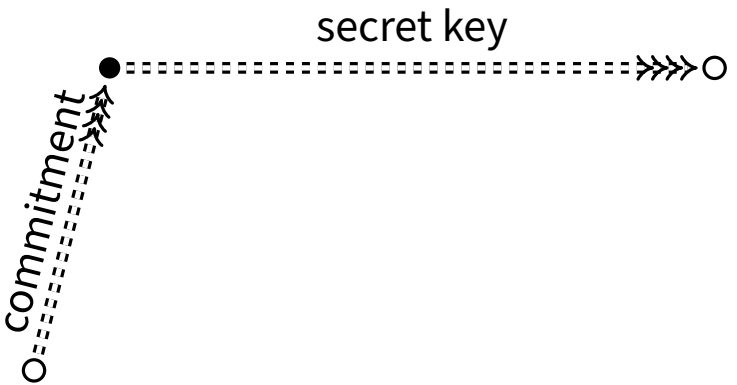
Φ_A

$$\deg(\varphi_A) \approx \deg(\Phi_A) = 2^x$$

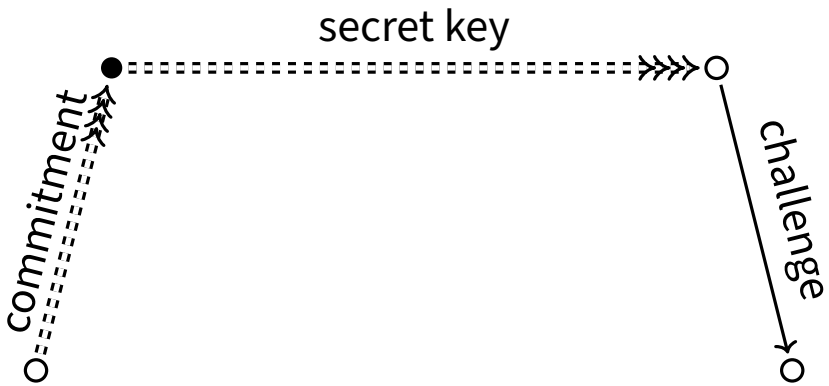


SQIsign2D

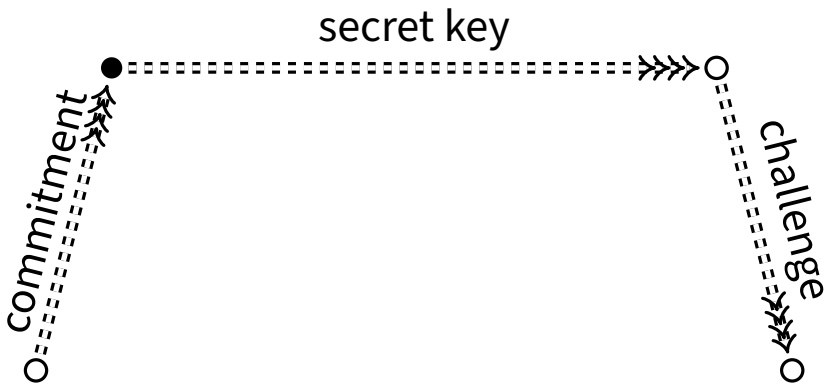
SQIsign2D



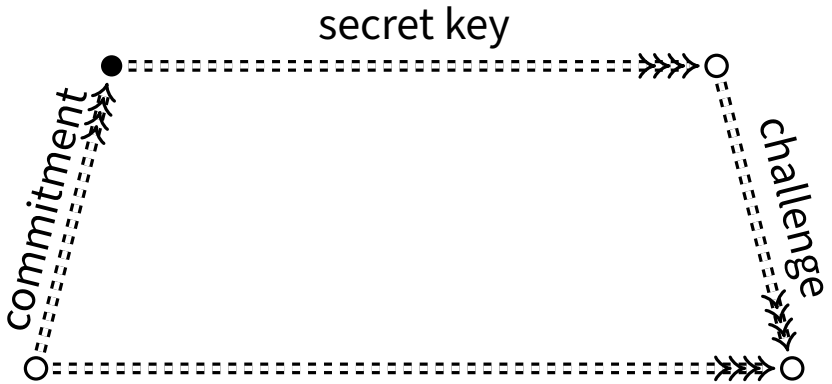
SQIsign2D



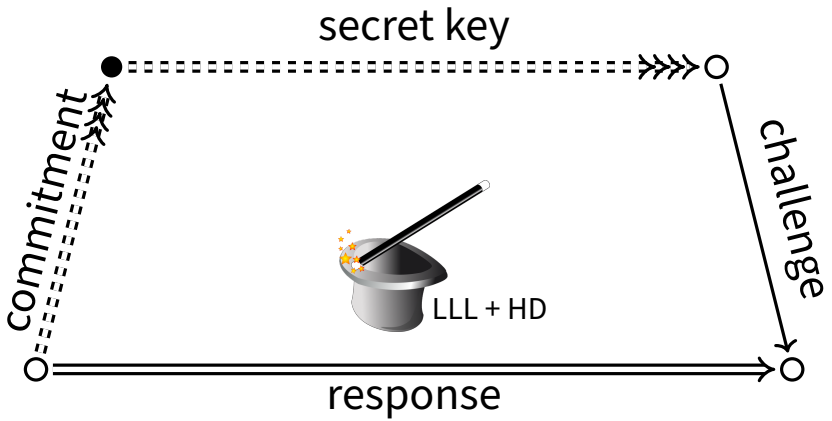
SQIsign2D



SQIsign2D



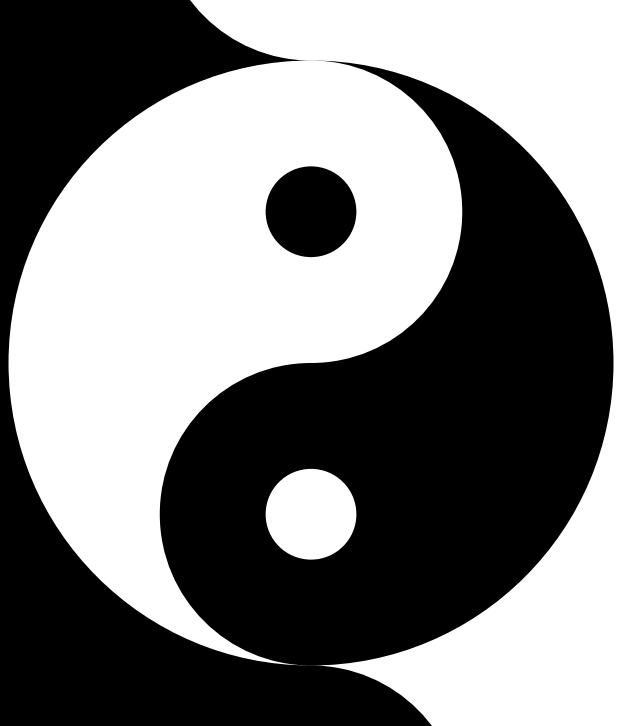
SQIsign2D



SQIsign2D

Bytes		Mcycles			
Public Key	Signature	Keygen	Sign	Verify	Security
66	148	60	160	9	NIST-1
98	222	170	460	29	NIST-3
130	294	360	940	62	NIST-5

CRYPTANALYSIS



CRYPTOGRAPHY