Isogeny-based Cryptography

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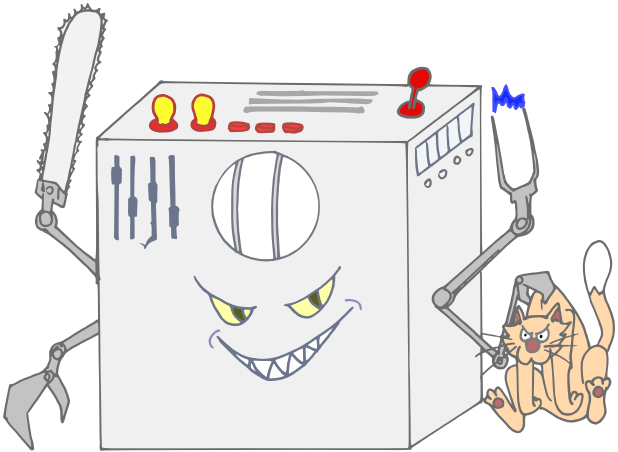
Are you quantum-safe?



Only two ways to Post-quantum Encryption

Noisy linear algebra:

Lattices, Codes

Far-fetched generalizations of the discrete logarithm:

Isogenies

What crypto from isogenies?

	Key exchange / Encryption	Identification / Signature	Other
Quadratic (Group actions)	Couveignes–Rostovtsev–Stolbunov CSIDH SCALLOP	SeaSign CSI-FiSh PEGASIS	Threshold PAKE …
Quaternionic	—	SQIsign SIDH-like signatures	Ring signatures Adaptor signatures Chameleon hash …
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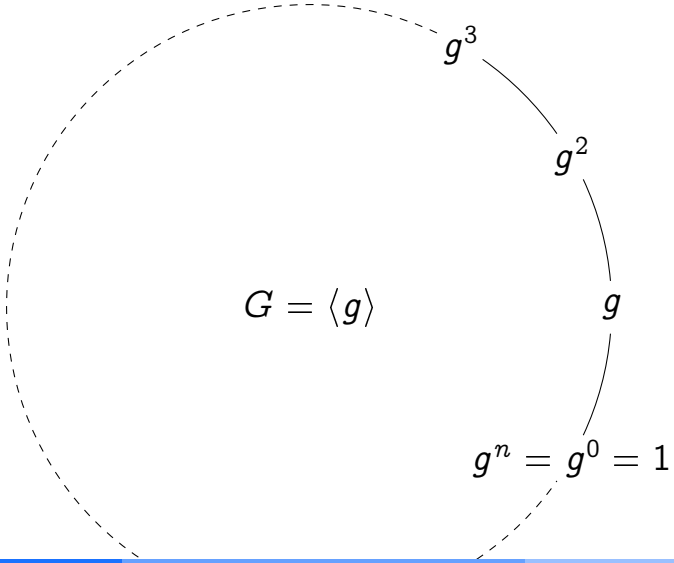
Cryptographic Group Actions

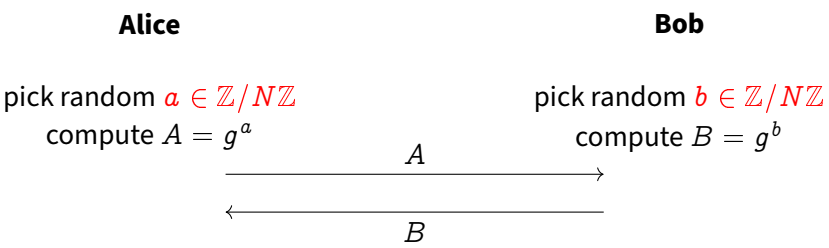
A cyclic group



Diffie–Hellman key exchange

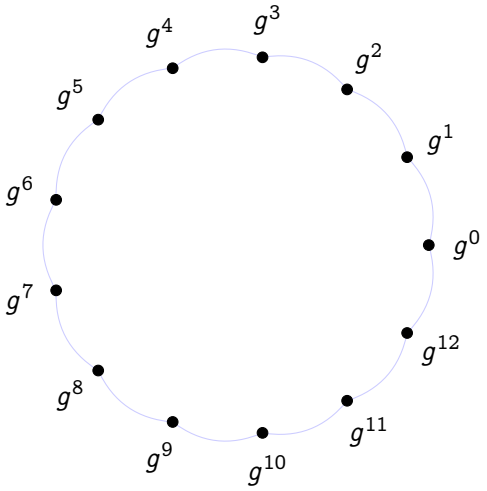
Goal: Alice and Bob have never met before. They are chatting over a public channel, and want to agree on a shared secret to start a private conversation.

Setup: They agree on a (large) cyclic group $G = \langle g \rangle$ of order N.



Shared secret is $B^a = g^{ab} = A^b$

What's needed for key exchange?



The axioms of a dlog group:

prod: ~~$g^a g^b = g^{a+b}$~~,

exp: $(g^a)^n = g^{na}$.

The hard problem:

dlog: $g^a \mapsto a$.

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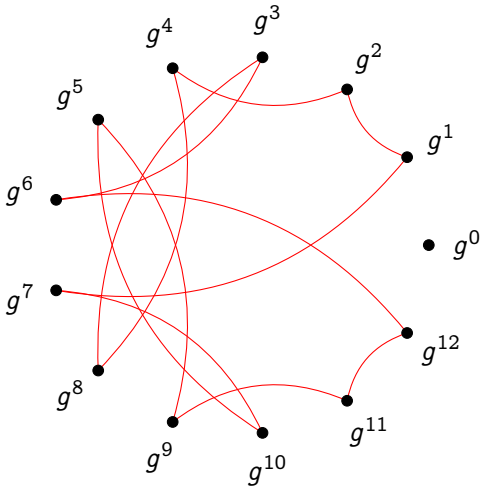

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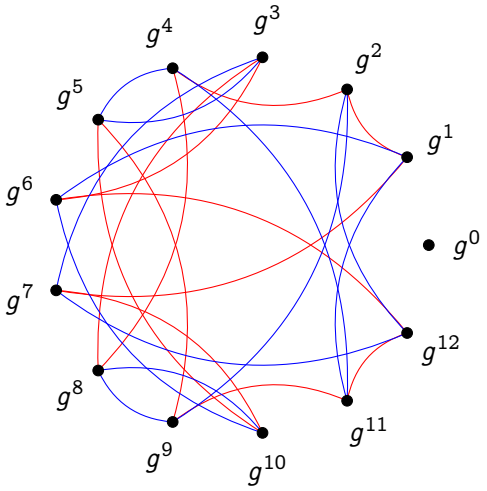

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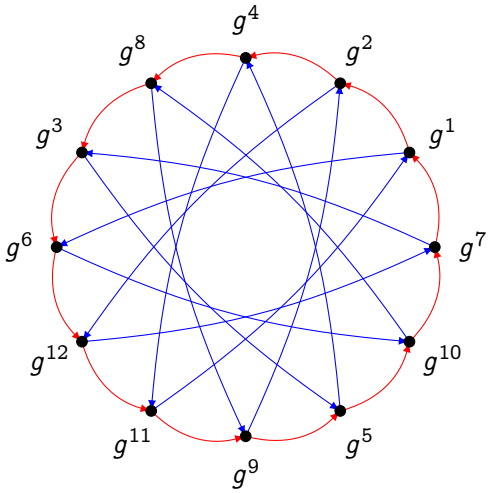

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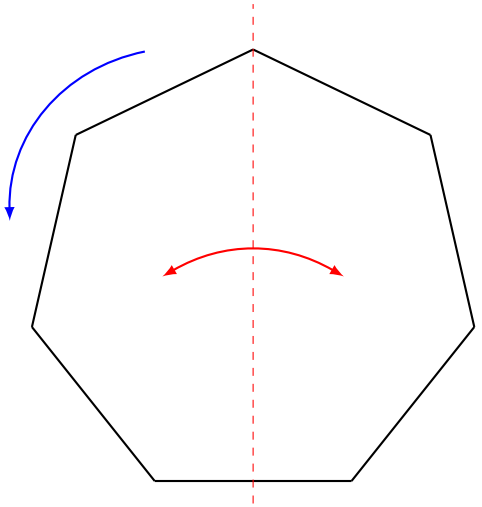
Symmetries = Group action



Group action

$\mathcal{G} \circlearrowright \mathcal{E}$: A (finite) set $\mathcal{E}$ acted upon by a group $\mathcal{G}$ freely and transitively:

$$\begin{aligned} * : \mathcal{G} \times \mathcal{E} &\longrightarrow \mathcal{E} \\ \mathfrak{g} * E &\longmapsto E' \end{aligned}$$

Compatibility: $\mathfrak{g}' * (\mathfrak{g} * E) = (\mathfrak{g}'\mathfrak{g}) * E$ for all $\mathfrak{g}, \mathfrak{g}' \in \mathcal{G}$ and $E \in \mathcal{E}$;

Identity: $\mathfrak{e} * E = E$ if and only if $\mathfrak{e} \in \mathcal{G}$ is the identity element;

Regularity: for all $E, E' \in \mathcal{E}$ there exist a unique $\mathfrak{g} \in \mathcal{G}$ such that $\mathfrak{g} * E' = E$.

Cryptographic Group Actions (Alamati, D., Montgomery, Patranabis 2021)

Hard Homogeneous Space (HHS) — Couveignes 1997 (eprint:2006/291)

$\mathcal{G} \circlearrowright \mathcal{E}$ such that $\mathcal{G}$ is commutative and:

- Evaluating $E' = \mathfrak{g} * E$ is easy;
- Inverting the action is hard.

Example

Let G be a group of order 13, then $(\mathbb{Z}/13\mathbb{Z})^\times \circlearrowright G$ defined by

$$a * g := g^a$$

is an HHS…

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Example

Let G be a group of order 13, then $(\mathbb{Z}/13\mathbb{Z})^\times \circlearrowright G$ defined by

$$a * g := g^a$$

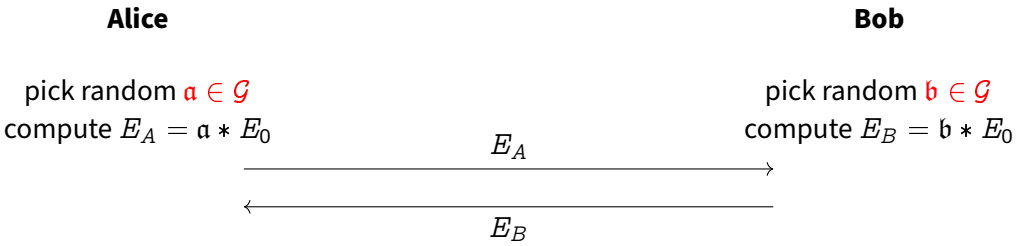
is an HHS… But

$$g^a \cdot g^b = g^{a+b}$$

has no interpretation as a group action!

Key exchange from group actions

Public parameters: A HHS $\mathcal{G} \circlearrowright \mathcal{E}$ of order N (large, but not necessarily prime).



Shared secret is $\mathfrak{a} * E_B = (\mathfrak{a}\mathfrak{b}) * E_0 = \mathfrak{b} * E_A$

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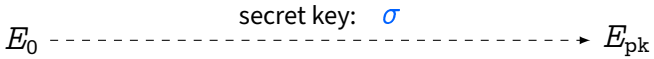
Identification scheme (Goldreich et al. 1992, Couveignes 1997, Stolbunov 2008)



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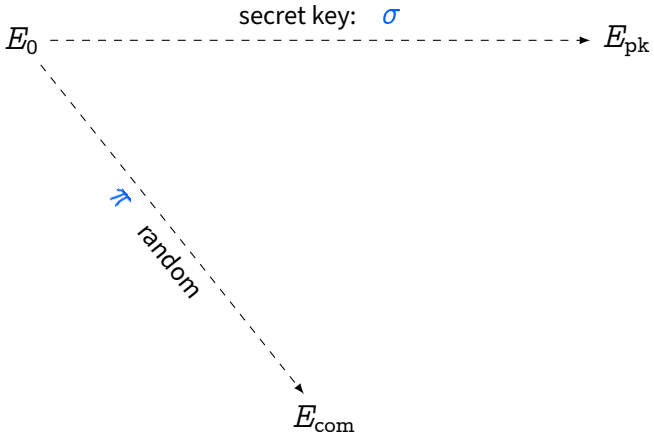

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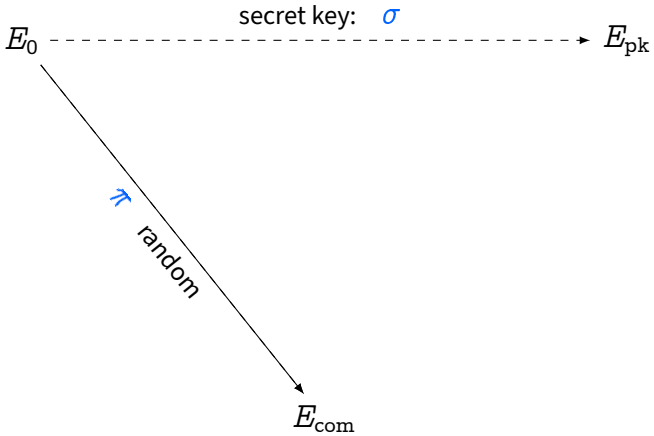

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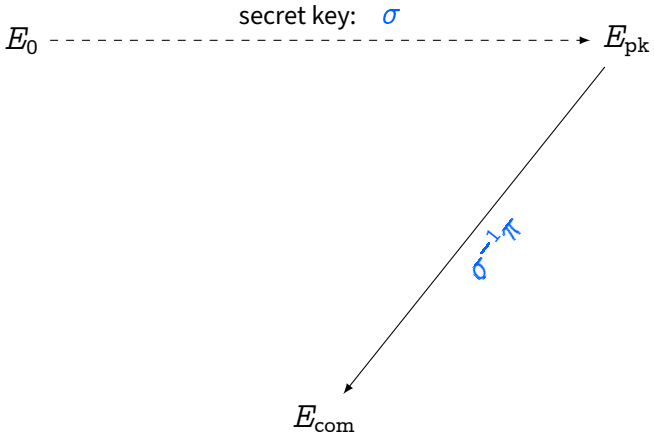


Quantum security

Fact: Shor's algorithm does not apply to Diffie-Hellman protocols from group actions.

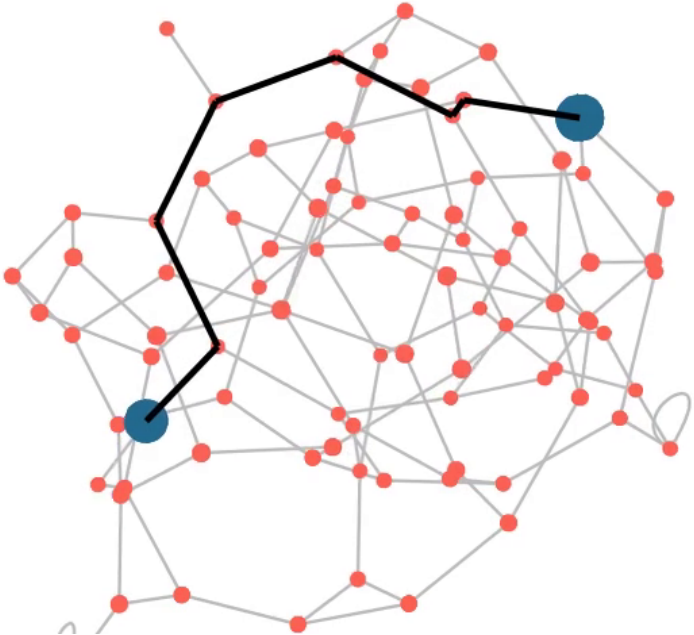
Subexponential attack $\exp(\sqrt{\log N \log\log N})$

- Reduction to the hidden shift problem by evaluating the group action in quantum superposition (subexponential cost);
- Well known reduction from the hidden shift to the dihedral (non-abelian) hidden subgroup problem;
- Kuperberg's algorithm solves the dHSP with a subexponential number of class group evaluations.
- Analyses suggest that 2^{64}-qbit security may be achieved somewhere around $\log N \approx 2048$.

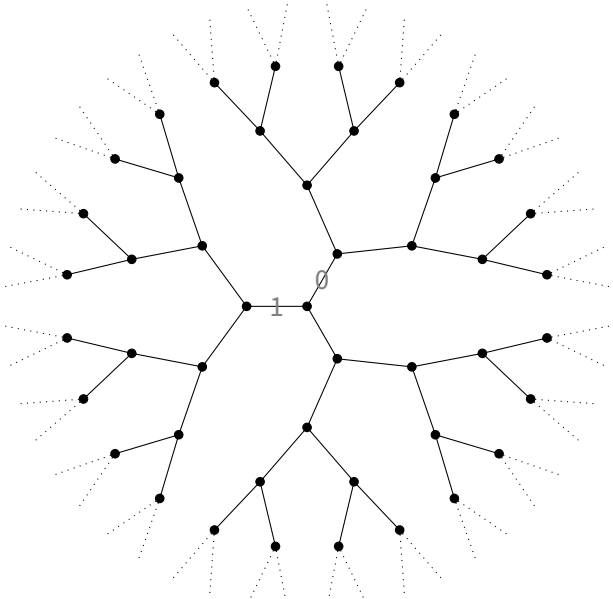
Expander graphs

The supersingular ℓ-isogeny graph

- Parameterized by primes p and $\ell \neq p$;
- $\approx p/12$ vertices;
- $(\ell + 1)$-regular;
- Connected;
- Optimal expander (in the sense of Ramanujan):
  - small diameter,
  - locally tree-like,
  - good mixing.



Random walks and Hash functions



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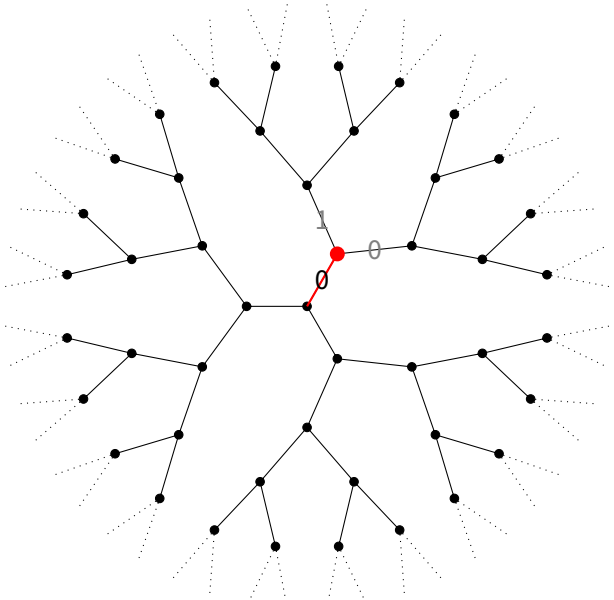

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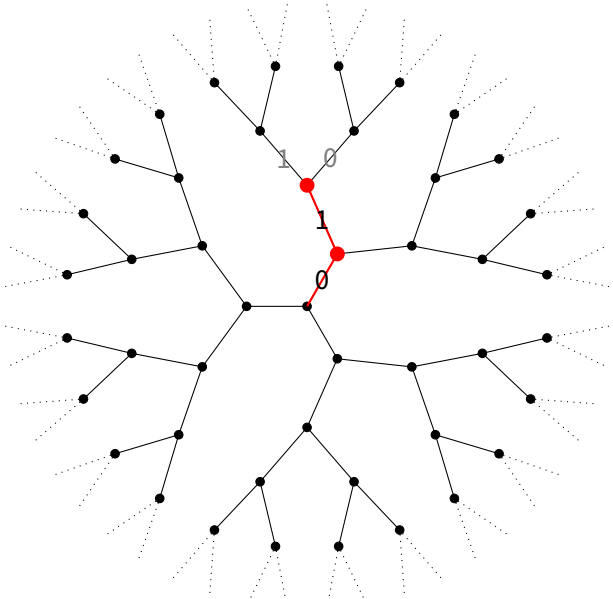

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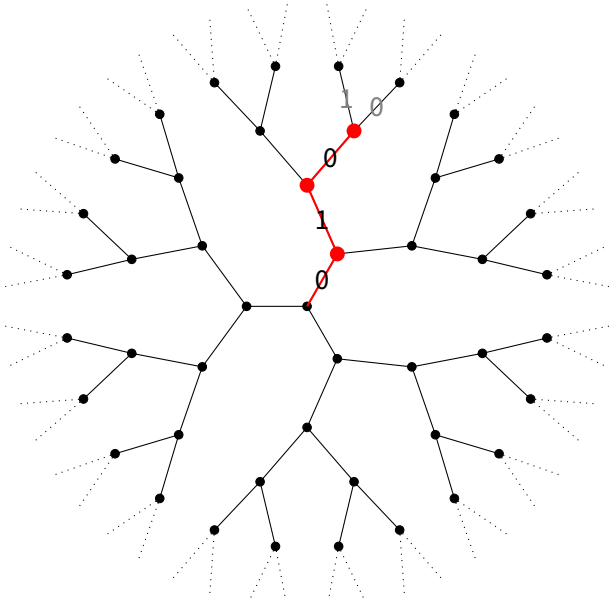

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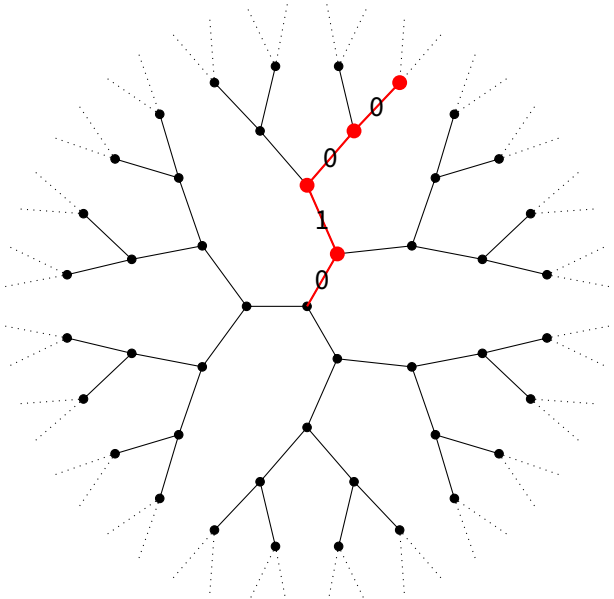
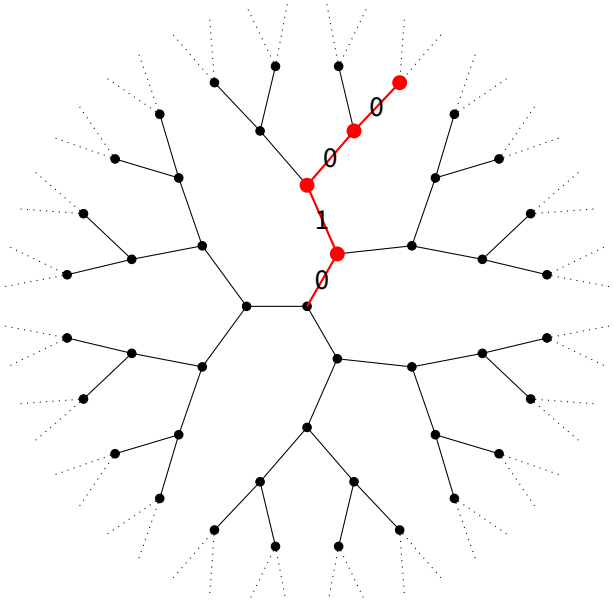


Random walks and Hash functions

Collisions: finding cycles

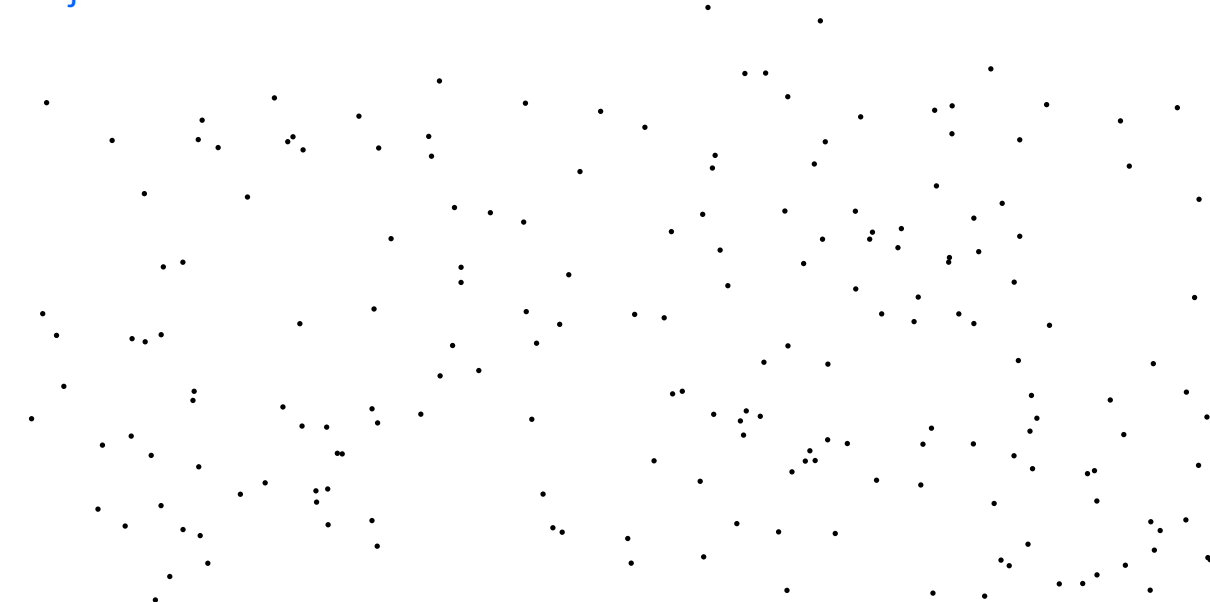
Preimages: finding paths



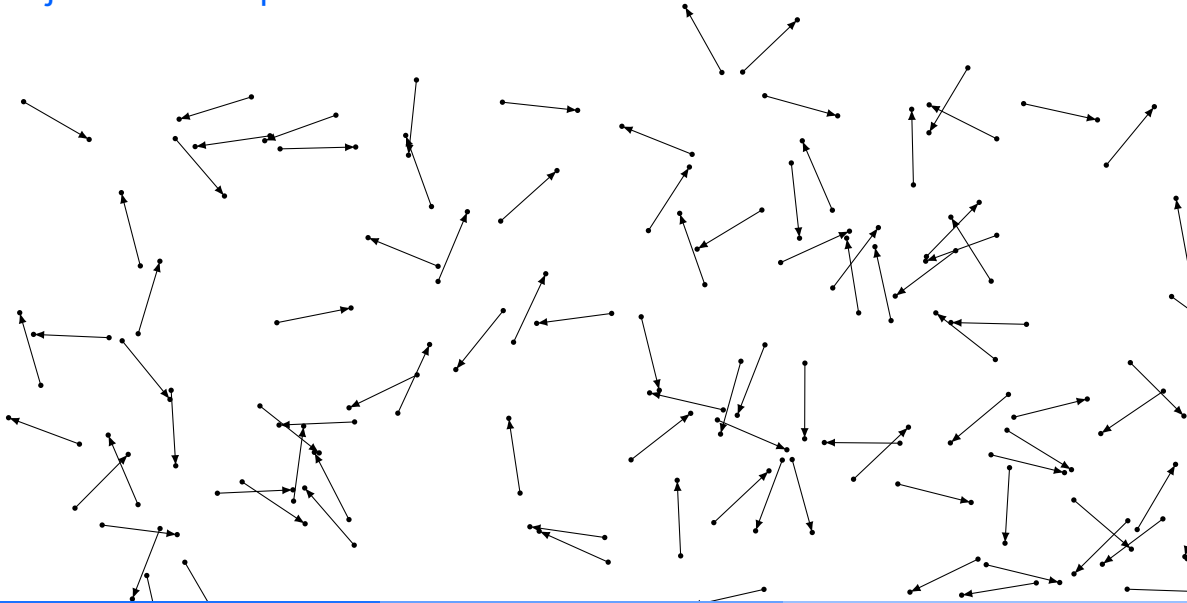
Forensic categories

(joint work with Basso, Patranabis, Radulescu, Wesolowski)

Objects

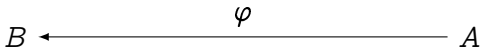


Objects and Morphisms



More than graphs…

- A finite set of objects $\mathcal{E}$,
- For each pair $A, B \in \mathcal{E}$ a set of morphisms $\mathrm{Hom}(A, B)$



$$\varphi \in \mathrm{Hom}(A, B)$$

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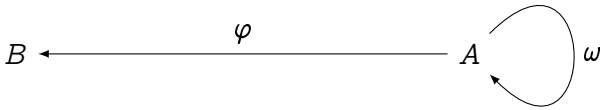
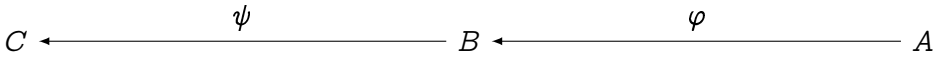


$\varphi \in \mathrm{Hom}(A, B)$ $\omega \in \mathrm{Hom}(A, A)$

Transitively closed



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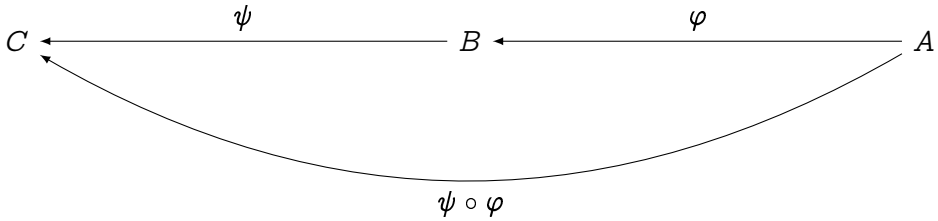
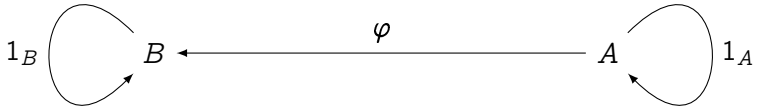


Identity morphisms



$$1_B \circ \varphi = \varphi = \varphi \circ 1_A$$

Computational axioms

- Every object and every morphism has a unique representation as a string;
- The representation of a morphism $\phi : A \to B$ identifies A and B;
- There exists efficient algorithms to verify that a string represents an object/morphism;
- There exists an origin object $O \in \mathcal{E}$ whose representation is known;
- …

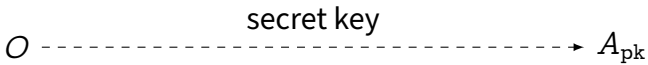
Axiom: *Walking*

Efficient algorithm for:

Input: Object A

Output:
- Uniformly random object B,
- Random morphism $\phi : A \to B$.

Identification scheme?



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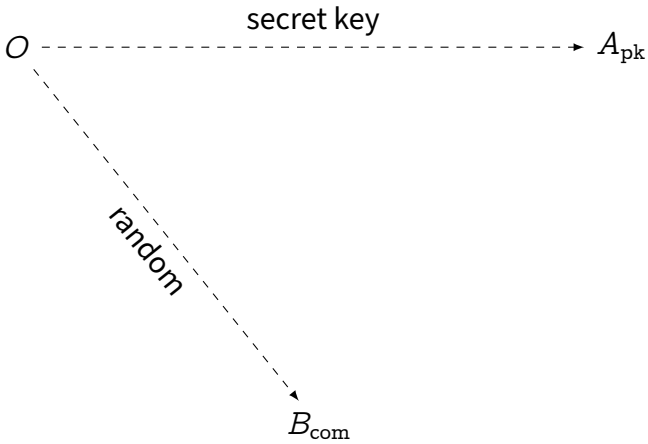

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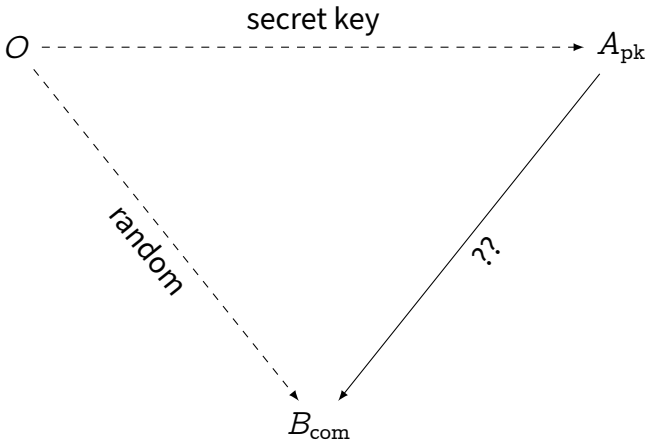
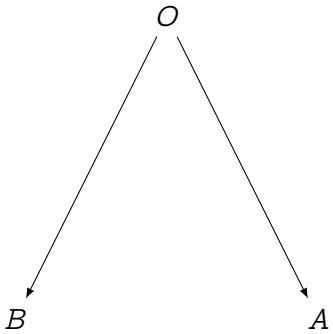


Axiom: Triangularizable



Input

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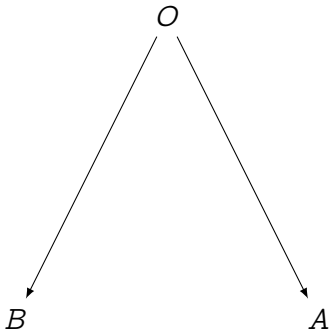

Input

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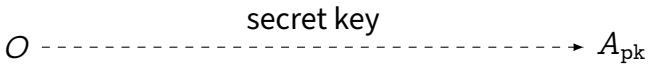

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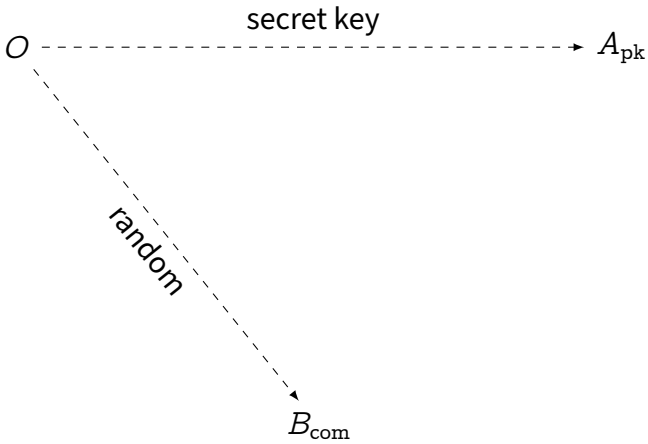

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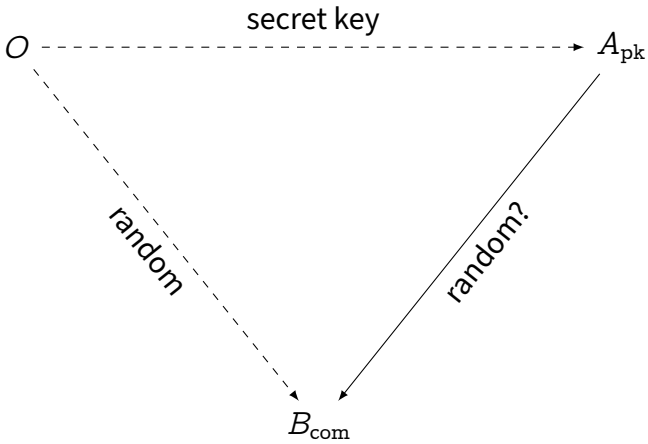
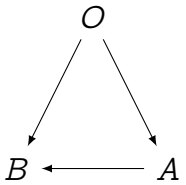


Hardness Assumption: *Walk-indistinguishable*

These two distributions are indistinguishable:

- Given A,
- *Walk* $\phi : A \to B$,



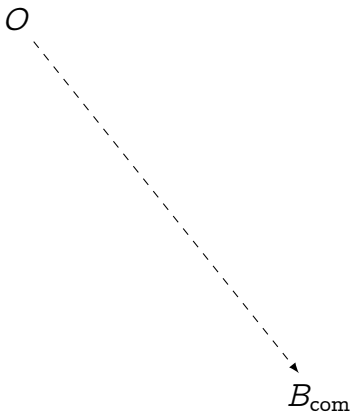
- Given $O \to A$,
- Sample random $O \to B$,
- *Triangularize* $\phi : A \to B$

Identification scheme?

O

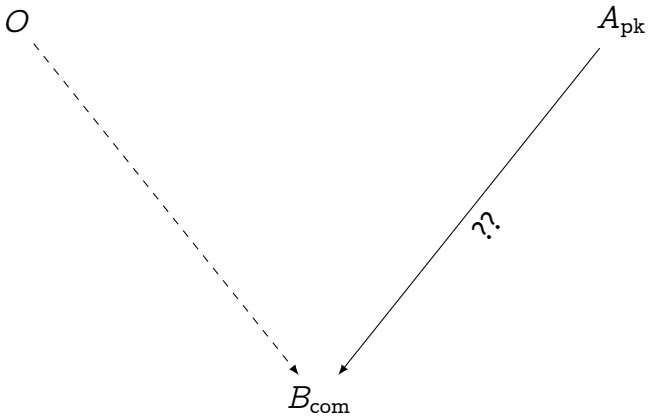
A_{pk}

Identification scheme?



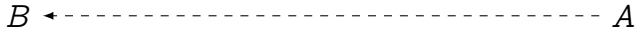
A_{pk}

Identification scheme?



Hardness assumption: *Morphism Finding*

Given two objects A, B, it is hard to find a morphism between them:



Summary

- *Effective* category,
- Origin object O,
- Walking,
- Triangularizable,
- Walk-indistinguishable,
- Hard to find morphisms.

GMW-style Identification scheme

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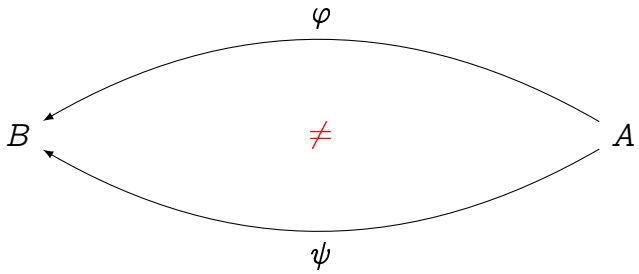
GMW-style Identification scheme

Lame!

Towards a stronger hardness assumption



Fingerprints

- A finite set $\mathcal{M}$ with at least 2 elements
- Partial maps for any pair A, B:

$$\mathrm{fp} \quad : \quad \mathrm{Hom}(A, B) \quad \rightarrow \quad \mathcal{M} \cup \{\perp\}$$

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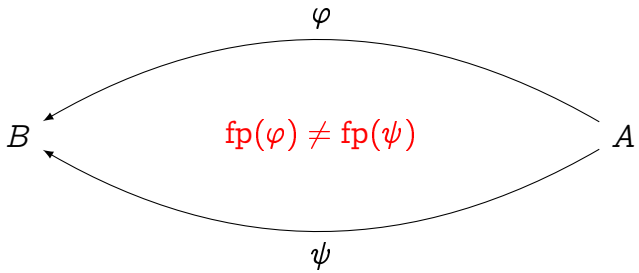
- Random walk $\to$ random fingerprint

Hardness assumption: Forensic hardness

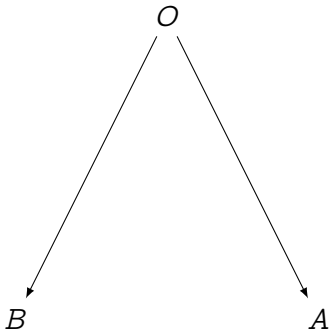
It is hard to find a pair of morphisms $A \to B$ with distinct fingerprints:



Easy corollary

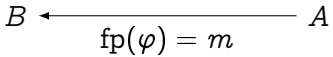
Forensic hardness $\Rightarrow$ Hard to find morphisms

Axiom: Triangularizable (reloaded)



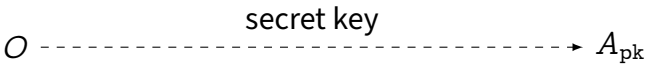
$m \in \mathcal{M}$

Input

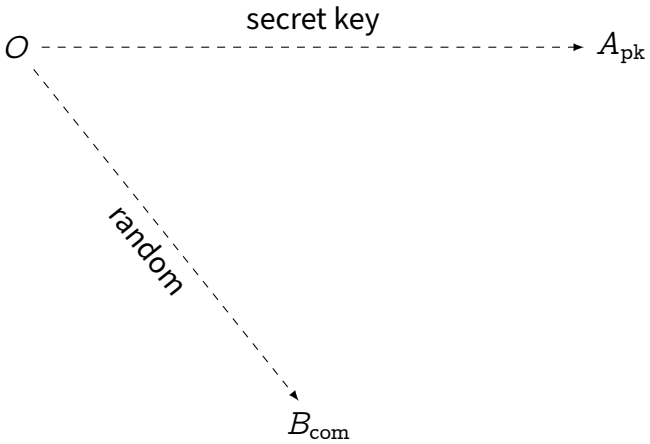


Output

SQIsign (in Forensic categories)

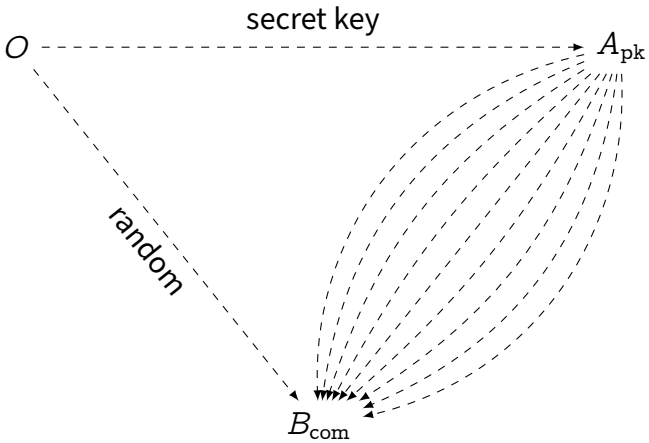


SQIsign (in Forensic categories)



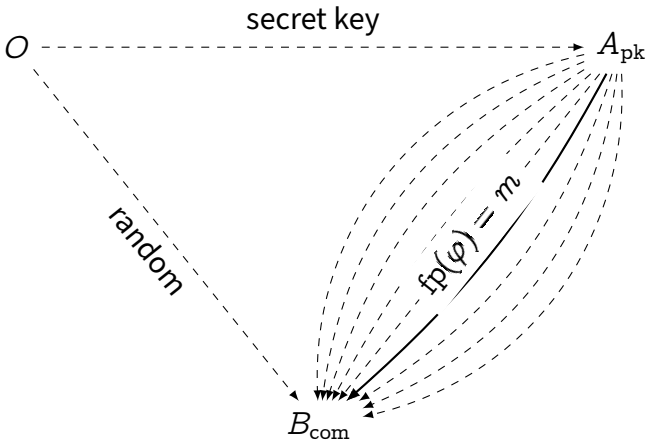
P: generate random commitment B_{com}

SQIsign (in Forensic categories)



P: generate random commitment B_{com}

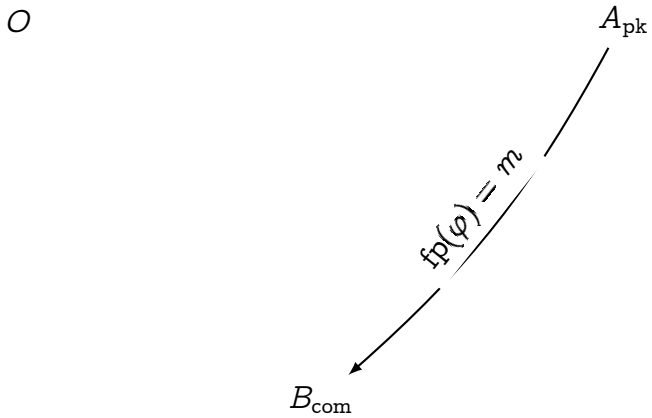
SQIsign (in Forensic categories)



P: generate random commitment B_{com}

V: challenge with random $m \in \mathcal{M}$

SQIsign (in Forensic categories)



P: generate random commitment B_{com}

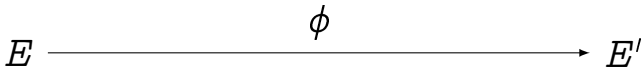
V: challenge with random $m \in \mathcal{M}$

P: respond with $\varphi : A_{\mathrm{pk}} \to B_{\mathrm{com}}$ such that $\mathrm{fp}(\varphi) = m$

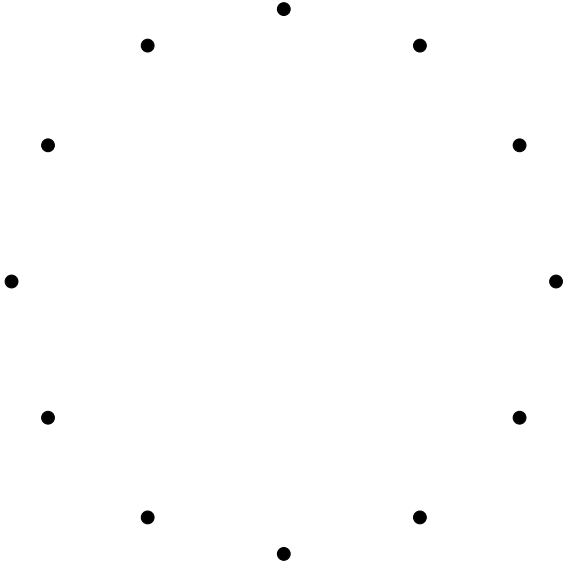
Isogenies, finally!

Isogenies = finite-kernel *algebraic* group morphisms: $E \to E'$

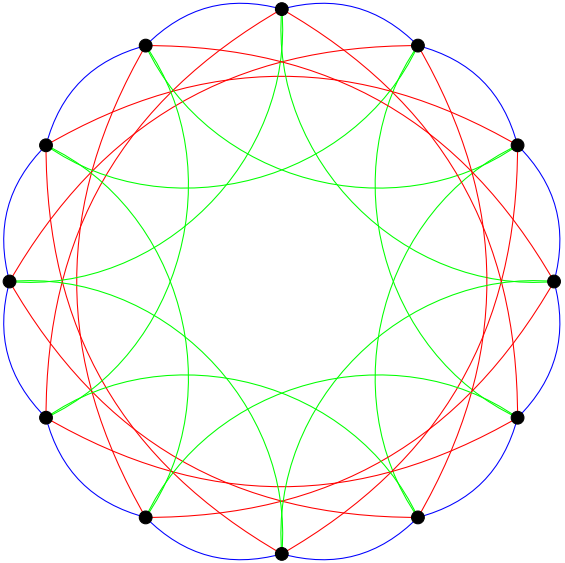
Endomorphisms = isogenies $E \to E$


E
ϕ
E'

Isogenous



Isogeny class



Isogeny class

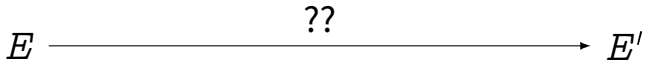
Isogeny graph

The isogeny problem

E

E'

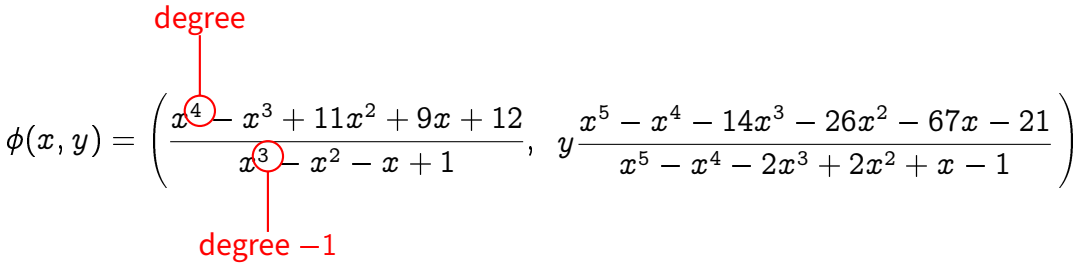
The isogeny problem



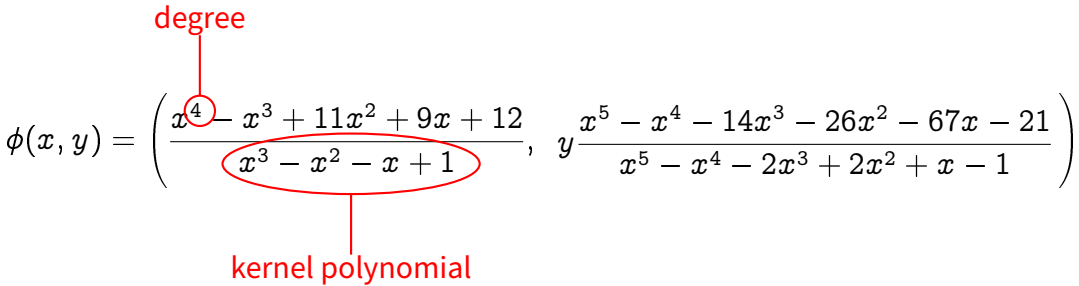
Anatomy of an isogeny

$$\phi(x,y) = \left(\frac{x^4 - x^3 + 11x^2 + 9x + 12}{x^3 - x^2 - x + 1},\ y\frac{x^5 - x^4 - 14x^3 - 26x^2 - 67x - 21}{x^5 - x^4 - 2x^3 + 2x^2 + x - 1} \right)$$

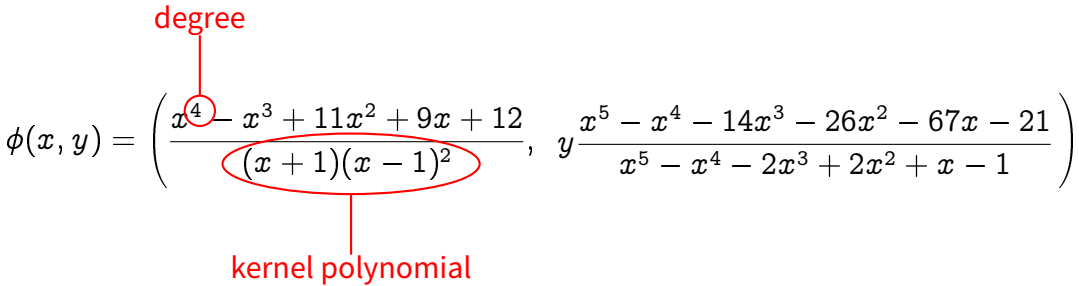
Anatomy of an isogeny



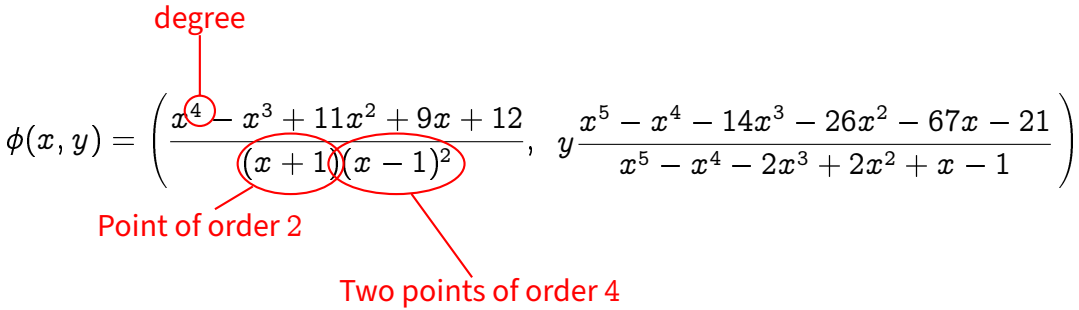
Anatomy of an isogeny



Anatomy of an isogeny



Anatomy of an isogeny



Anatomy of an isogeny

degree

$$\phi(x,y) = \left(\frac{x^4 - x^3 + 11x^2 + 9x + 12}{h(x)},\ y\frac{x^5 - x^4 - 14x^3 - 26x^2 - 67x - 21}{x^5 - x^4 - 2x^3 + 2x^2 + x - 1} \right)$$

$$h(x) = \prod_{P \in K \setminus \{0\}} \big(x - x(P)\big)$$

Anatomy of an isogeny

computed by Vélu–Elkies–Kohel formulas

$$\phi(x,y) = \left(\frac{g(x)}{h(x)}, \quad y\frac{x^5 - x^4 - 14x^3 - 26x^2 - 67x - 21}{x^5 - x^4 - 2x^3 + 2x^2 + x - 1} \right)$$

$$h(x) = \prod_{P \in K \setminus \{0\}} \big(x - x(P)\big)$$

Anatomy of an isogeny

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Anatomy of an isogeny

computed by Vélu–Elkies–Kohel formulas

$$\phi(x,y) = \left(\frac{g(x)}{h(x)}, \quad y\left(\frac{g(x)}{h(x)}\right)' \right)$$

$$h(x) = \prod_{P \in K \setminus \{0\}} \big(x - x(P)\big)$$

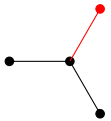
Input: Finite kernel $K \subset E$ of order d;

Output: Rational fractions $\phi(x,y)$;

Complexity: $\tilde{O}(d)$ operations.

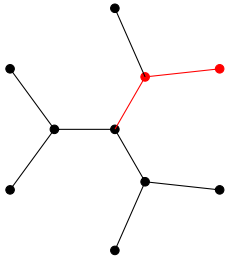
Isogeny graphs

$$\frac{x^2 + \cdots}{x + \cdots}$$



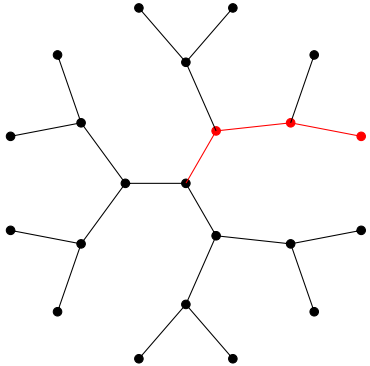
Isogeny graphs

$$\frac{x^2 + \cdots}{x + \cdots} \circ \frac{x^2 + \cdots}{x + \cdots}$$



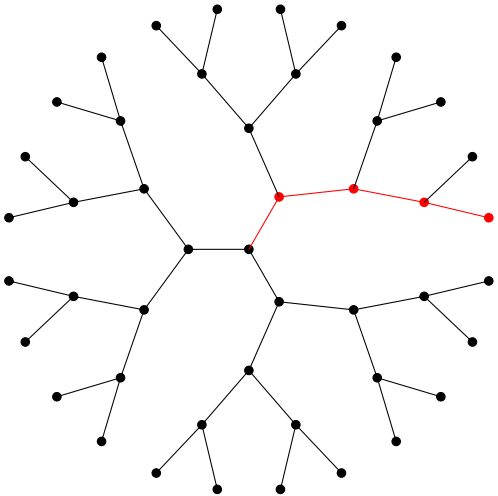
Isogeny graphs

$$\frac{x^2 + \cdots}{x + \cdots} \circ \frac{x^2 + \cdots}{x + \cdots} \circ \frac{x^2 + \cdots}{x + \cdots}$$



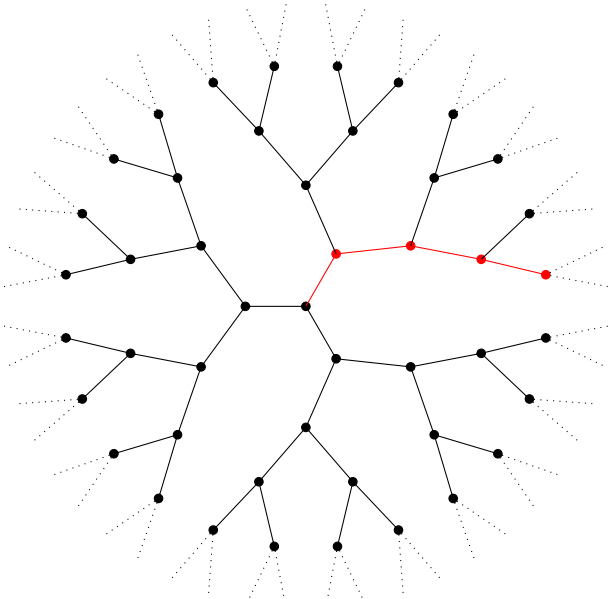
Isogeny graphs

$$\frac{x^2+\cdots}{x+\cdots} \circ \frac{x^2+\cdots}{x+\cdots} \circ \frac{x^2+\cdots}{x+\cdots} \circ \frac{x^2+\cdots}{x+\cdots}$$



Isogeny graphs

$$\frac{x^2+\cdots}{x+\cdots} \circ \frac{x^2+\cdots}{x+\cdots} \circ \frac{x^2+\cdots}{x+\cdots} \circ \frac{x^2+\cdots}{x+\cdots}$$



Two computational worlds

	Quadratic imaginary	Quaternionic
$\mathrm{rank}\,\mathrm{Hom}(E, E')$	2	4
Endomorphism algebra	number field	quaternion algebra
Maximal orders	one	many
Ideal class…	…group	…set
Find isogeny $E \to E'$	hard	hard
Convert isogenies $\leftrightarrow$ ideals	easy[1]	easy[1]
Compute $\mathrm{End}(E)$	easy	hard

[1] When $\mathrm{End}(E)$ is known

A cryptographic group action

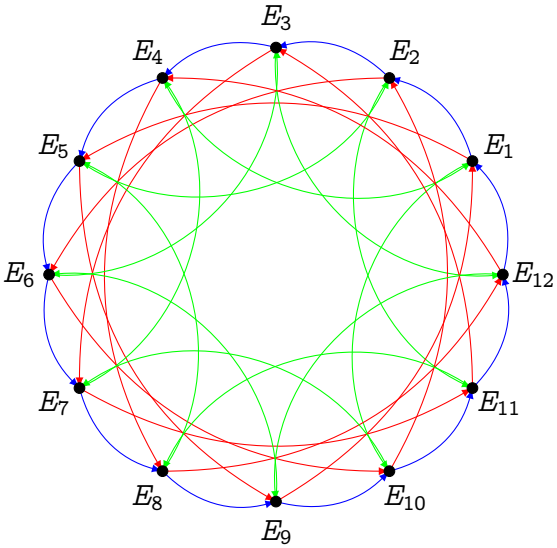
The quadratic imaginary case

Complex Multiplication

Fix an order $\mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$:

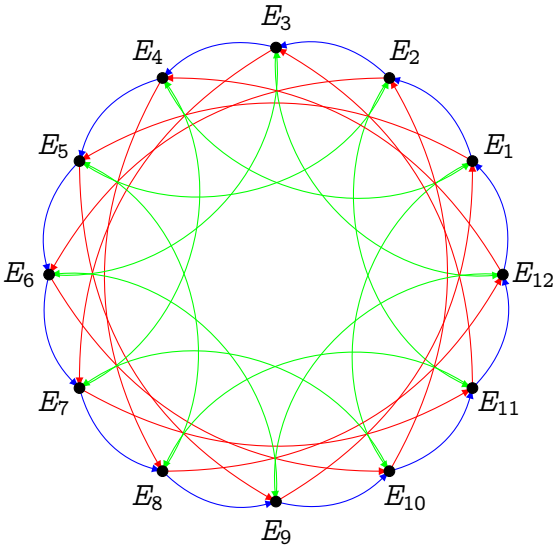
- Invertible ideal classes form a finite abelian group
- $\mathrm{Cl}(\mathcal{O})$ acts freely and transitively on the set of elliptic curves with $\mathrm{End}(E) \simeq \mathcal{O}$.

Class group action



Action $E_2 = E_1^{\bullet}$
$E_5 = E_1^{\bullet}$
$E_{10} = E_1^{\bullet}$

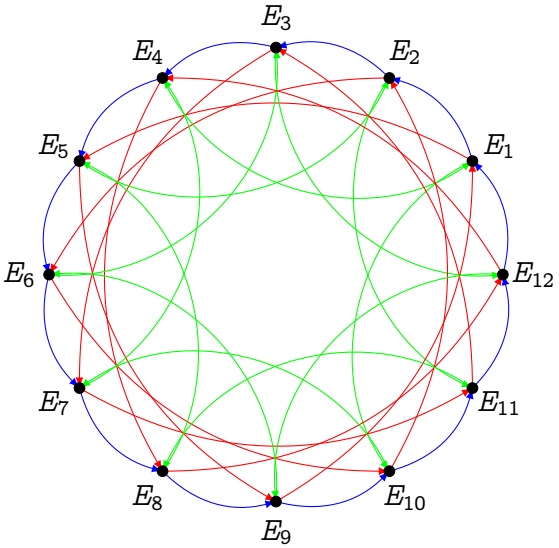
Class group action



Group $((E^{\bullet})^{\bullet})^{\bullet} = E^{(\bullet\bullet\bullet)} = E_1^{\bullet}$

Action $E_2 = E_1^{\bullet}$
$E_5 = E_1^{\bullet}$
$E_{10} = E_1^{\bullet}$

Class group action



Commutative $E^{(\bullet\bullet)} = E^{(\bullet\bullet)}$

Group $((E^{\bullet})^{\bullet})^{\bullet} = E^{(\bullet\bullet\bullet)} = E_1^{\bullet}$

Action $E_2 = E_1^{\bullet}$
$E_5 = E_1^{\bullet}$
$E_{10} = E_1^{\bullet}$

Two computational worlds

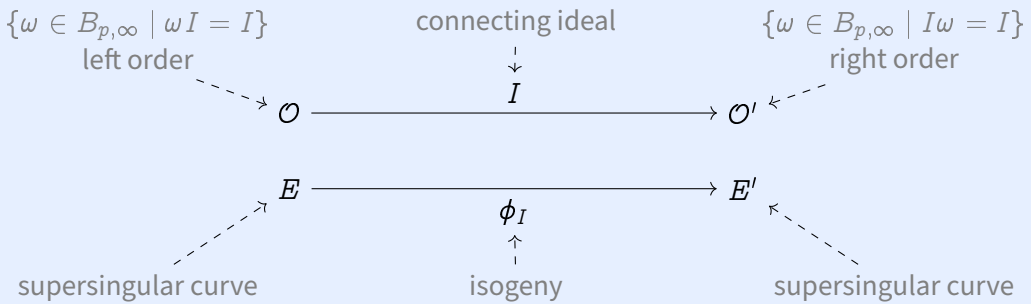
	Quadratic imaginary	Quaternionic
$\mathrm{rank}\,\mathrm{Hom}(E, E')$	2	4
Endomorphism algebra	number field	quaternion algebra
Maximal orders	one	many
Ideal class…	…group	…set
Find isogeny $E \to E'$	hard	hard
Convert isogenies $\leftrightarrow$ ideals	easy[1]	easy[1]
Compute $\mathrm{End}(E)$	easy	hard

[1] When $\mathrm{End}(E)$ is known

A forensic category

The Deuring correspondence (for supersingular curves)

An equivalence of categories (roughly)



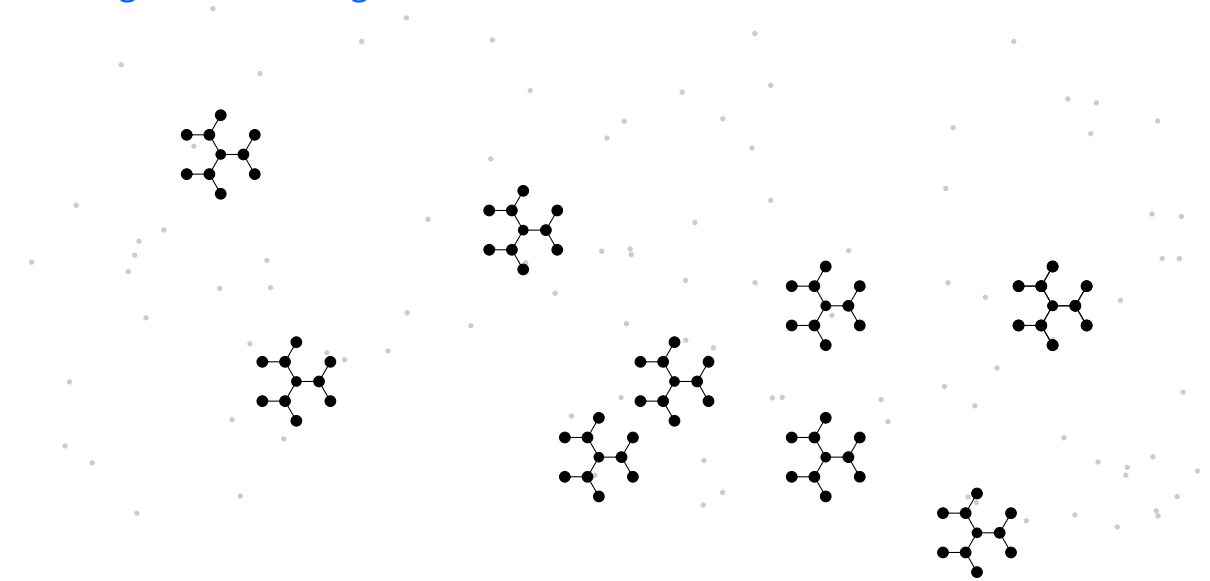
The endomorphism ring problem

Given a random supersingular curve E, compute $\mathrm{End}(E)$

Contagious knowledge

Contagious knowledge

Contagious knowledge



$\mathrm{End}(E)$ as a trapdoor

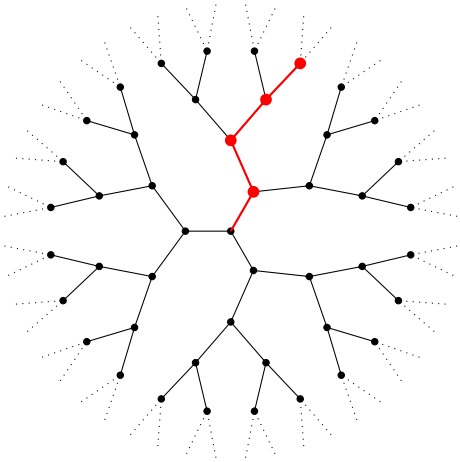
Input	Output	
random E	$\mathrm{End}(E)$	hard
random E	$\omega \in \mathrm{End}(E)$	hard
random E	$\phi : E_0 \to E$	hard
$\mathrm{End}(E)$	E	easy
$\mathrm{End}(E), \mathrm{End}(E')$	connecting ideal	easy
$I \subset \mathrm{End}(E)$	$\phi_I : E \to E'$	easy
$\mathrm{End}(E), \quad \phi : E \to E'$	$I_\phi \subset \mathrm{End}(E), \quad \mathrm{End}(E')$	easy

Axiom: *Walking*

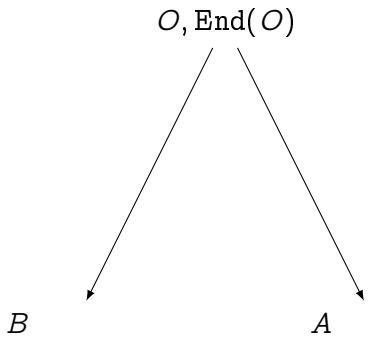
Efficient algorithm for:

Input: Curve A

Output:
- Uniformly random curve B,
- Random isogeny $\phi : A \to B$.

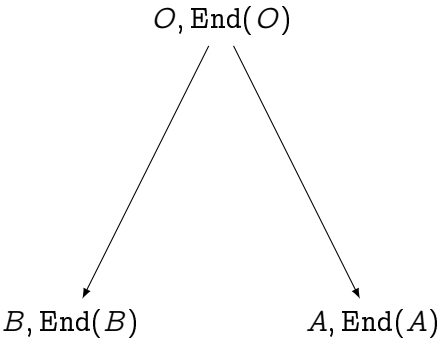


Axiom: Triangularizable



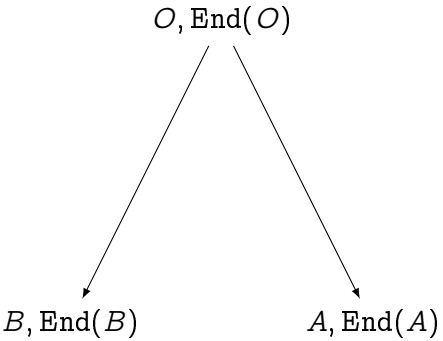
Input

Axiom: Triangularizable

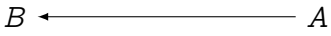


Input

Axiom: Triangularizable

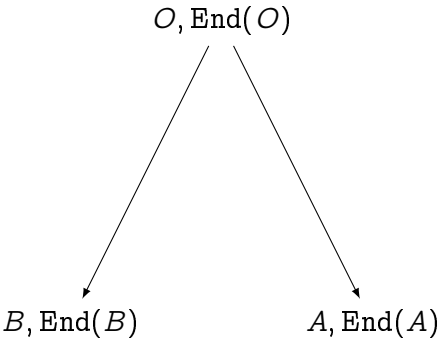


Input



Output

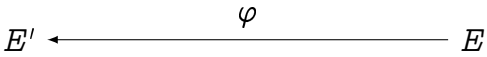
Axiom: Triangularizable



Input

Output

Fingerprints



How the isogeny acts on points of order N:

Kernel fingerprint: $\ker \varphi \cap E[N]$

(OG SQIsign)

Level fingerprint: Action on fixed bases of $E[N]$ and $E'[N]$

(joint with Borin, Lido, Schaeffler)

Worth the effort?

SQIsign v2 (June 2025)

	Sizes (bytes)		Timings (ms)		
Security	Public Key	Signature	Keygen	Sign	Verify
NIST-1	66	148	25	60	3
NIST-3	98	222	67	161	9
NIST-5	130	294	119	300	18

What crypto from isogenies?

	Key exchange / Encryption	Identification / Signature	Other
Quadratic (Group actions)	Couveignes–Rostovtsev–Stolbunov CSIDH SCALLOP	SeaSign CSI-FiSh PEGASIS	Threshold PAKE …
Quaternionic (Forensic cat.)	—	SQIsign SIDH-like signatures	Ring signatures Adaptor signatures Chameleon hash …
Ad hoc (Level structures)	SIDH † SIDH fixes FESTA	SIDH-like signatures	Time-release crypto …

Thank you

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